

Dynamical Large Deviations for a Boundary Driven Stochastic Lattice Gas Model with Many Conserved Quantities

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Received: 20 November 2009 / Accepted: 29 March 2010 / Published online: 3 April 2010
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Abstract We prove the dynamical large deviations for a particle system in which particles may have different velocities. We assume that we have two infinite reservoirs of particles at the boundary: this is the so-called boundary driven process. The dynamics we considered consists of a weakly asymmetric simple exclusion process with collision among particles having different velocities.

Keywords Boundary driven exclusion processes · Large deviations

1 Introduction

In the last years there has been considerable progress in understanding stationary non equilibrium states: diffusive systems in contact with different reservoirs at the boundary imposing a gradient on the conserved quantities of the system. In these systems there is a flow of matter through the system and the dynamics is not reversible. The main difference with respect to equilibrium (reversible) states is the following: in equilibrium, the invariant measure, which determines the thermodynamic properties, is given for free by the Gibbs distribution specified by the Hamiltonian; on the other hand, in non equilibrium states the construction of the stationary state requires the solution of a dynamical problem. One of

Research supported by CNPq.

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the most striking typical property of these systems is the presence of long-range correlations. For the symmetric simple exclusion this was already shown in a pioneering paper by Spohn [14]. We refer to [5, 7] for two recent reviews on this topic.

We discuss this issue in the context of stochastic lattice gases in a box of linear size N with birth and death processes at the boundary modeling the reservoirs. We consider the case when there are many thermodynamic variables: the local density denoted by ρ , and the local momentum denoted by $p_k, k = 1, \dots, d, d$ being the dimension of the box.

Let the set of possible velocities, \mathcal{V} , be a finite subset of \mathbb{R}^d , and for a point $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, let $\tilde{x} = (x_2, \dots, x_d)$. The model which we will study can be informally described as follows: fix a velocity $v \in \mathcal{V}$, an integer $N \geq 1$, and boundary densities $0 < \alpha_v(\cdot) < 1$ and $0 < \beta_v(\cdot) < 1$; at any given time, each site of the set $\{1, \dots, N - 1\} \times \{0, \dots, N - 1\}^{d-1}$ is either empty or occupied by one particle at velocity v . In the bulk, each particle attempts to jump at any of its neighbors at the same velocity, with a weakly asymmetric rate. To respect the exclusion rule, the particle jumps only if the target site at the same velocity v is empty; otherwise nothing happens. At the boundary, sites with first coordinates given by 1 or $N - 1$ have particles being created or removed in such a way that the local densities are $\alpha_v(\tilde{x})$ and $\beta_v(\tilde{x})$: at rate $\alpha_v(\tilde{x}/N)$ a particle is created at $\{1\} \times \{\tilde{x}\}$ if the site is empty, and at rate $1 - \alpha_v(\tilde{x})$ the particle at $\{1\} \times \{\tilde{x}\}$ is removed if the site is occupied, and at rate $\beta_v(\tilde{x})$ a particle is created at $\{N - 1\} \times \{\tilde{x}\}$ if the site is empty, and at rate $1 - \beta_v(\tilde{x})$ the particle at $\{N - 1\} \times \{\tilde{x}\}$ is removed if the site is occupied. Superposed to this dynamics, there is a collision process which exchange velocities of particles in the same site in a way that momentum is conserved. Similar models have been studied by [1, 8, 11]. In fact, the model we consider here is based on the model of Esposito et al. [8] which was used to derive the Navier-Stokes equation. It is also noteworthy that the derivation of hydrodynamic limits and macroscopic fluctuation theory for a system with two conserved quantities have been studied in [4].

The hydrodynamic limit for the above model has been proved in [12]. The hydrodynamic equation is derived from the underlying stochastic dynamics through an appropriate scaling limit in which the microscopic time and space coordinates are rescaled diffusively. The hydrodynamic equation thus represents the law of large numbers for the empirical density of the stochastic lattice gas. The convergence has to be understood in probability with respect to the law of the stochastic lattice gas. Once it is established, a natural question is to consider large deviations from the hydrodynamic limit.

This article thus provides a derivation of the dynamical large deviations for this model. As usual, the main difficulty appears in the proof of the lower bound where one needs to show that any trajectory $\lambda_t, 0 \leq t \leq T$, with finite rate function, $I_T(\lambda) < \infty$, can be approximated by a sequence of regular trajectories $\{\lambda^n : n \geq 1\}$ such that

$$\lambda^n \rightarrow \lambda \quad \text{and} \quad I_T(\lambda^n) \rightarrow I_T(\lambda). \tag{1.1}$$

To avoid this difficulty, we follow the method introduced in [9]. It is well known that if $I_T(\lambda) < \infty$, then there exists an external field H associated to λ , in the sense that λ solves a hydrodynamic equation perturbed by the external field H . The strategy of [9] is to approximate the external field H by a sequence of smooth functions, H_n , and then to show that the corresponding weak solutions of the hydrodynamical equations perturbed by H_n converge to λ in the sense (1.1).

The main difference of our proof with respect to theirs, is that their proof of the convergence (1.1) relied on some energy estimates that we were not able to achieve due to the presence of velocities. Therefore, we had to overcome this problem by taking an alternative approach at that part. More specific details are given in Sect. 5.

The article is organized as follows: in Sect. 2 we establish the notation and state the main results of the article; in Sect. 3, we review the hydrodynamics for this model, that was obtained in [12]; in Sect. 4, several properties of the rate function are derived; Sect. 5 proves the $I_T(\cdot|\gamma)$ -density, which is a key result for proving the lower bound; finally, in Sect. 6 the proofs of the upper and lower bounds of the dynamical large deviations are given.

2 Notation and Results

Fix a positive integer $d \geq 1$, and denote by D^d the open set $(0, 1) \times \mathbb{T}^{d-1}$, where \mathbb{T}^k is the k -dimensional torus $(\mathbb{R}/\mathbb{Z})^k = [0, 1)^k$, and by Γ the boundary of D^d : $\Gamma = \{(u_1, \dots, u_d) \in [0, 1] \times \mathbb{T}^{d-1}; u_1 = 0 \text{ or } 1\}$.

For an open subset Λ of $\mathbb{R} \times \mathbb{T}^{d-1}$, $C^m(\Lambda)$, $1 \leq m \leq +\infty$, stands for the space of m -continuously differentiable real functions defined on Λ . Let $C_0^m(\Lambda)$ (resp. $C_c^m(\Lambda)$), $1 \leq m \leq +\infty$, be the subset of functions in $C^m(\Lambda)$ which vanish at the boundary of Λ (resp. with compact support in Λ).

For each integer $N \geq 1$, denote by $\mathbb{T}_N^{d-1} = (\mathbb{Z}/N\mathbb{Z})^{d-1} = \{0, \dots, N - 1\}^{d-1}$, the discrete $(d - 1)$ -dimensional torus of length N . Let $D_N^d = \{1, \dots, N - 1\} \times \mathbb{T}_N^{d-1}$ be the cylinder in \mathbb{Z}^d of length $N - 1$ and basis \mathbb{T}_N^{d-1} and let $\Gamma_N = \{(x_1, \dots, x_d) \in \mathbb{Z} \times \mathbb{T}_N^{d-1}; x_1 = 1 \text{ or } (N - 1)\}$ be the boundary of D_N^d .

Let $\mathcal{V} \subset \mathbb{R}^d$ be a finite set of velocities $v = (v_1, \dots, v_d)$. Assume that \mathcal{V} is invariant under reflexions and permutations of the coordinates:

$$(v_1, \dots, v_{i-1}, -v_i, v_{i+1}, \dots, v_d) \quad \text{and} \quad (v_{\sigma(1)}, \dots, v_{\sigma(d)}) \tag{2.1}$$

belong to \mathcal{V} for all $1 \leq i \leq d$, and all permutations σ of $\{1, \dots, d\}$, provided (v_1, \dots, v_d) belongs to \mathcal{V} .

On each site of D_N^d , at most one particle for each velocity is allowed. We denote: the number of particles with velocity v at x , $v \in \mathcal{V}$, $x \in D_N^d$, by $\eta(x, v) \in \{0, 1\}$; the number of particles in each velocity v at a site x by $\eta_x = \{\eta(x, v); v \in \mathcal{V}\}$; and a configuration by $\eta = \{\eta_x; x \in D_N^d\}$. The set of particle configurations is $X_N = (\{0, 1\}^{\mathcal{V}})^{D_N^d}$.

On the interior of the domain, the dynamics consists of two parts: (i) each particle of the system evolves according to a nearest neighbor weakly asymmetric random walk with exclusion among particles of the same velocity, and (ii) binary collision between particles of different velocities. Let $p(x, v)$ be an irreducible probability transition function of finite range, and mean velocity v :

$$\sum_x xp(x, v) = v.$$

The jump law and the waiting times are chosen so that the jump rate from site x to site $x + y$ for a particle with velocity v is

$$P_N(y, v) = \frac{1}{2} \sum_{j=1}^d (\delta_{y, e_j} + \delta_{y, -e_j}) + \frac{1}{N} p(y, v),$$

where $\delta_{x,y}$ stands for the Kronecker delta, which equals one if $x = y$ and 0 otherwise, and $\{e_1, \dots, e_d\}$ is the canonical basis in \mathbb{R}^d .

2.1 The Boundary Driven Exclusion Process

Our main interest is to examine the stochastic lattice gas model given by the generator \mathcal{L}_N which is the superposition of the boundary dynamics with the collision and exclusion:

$$\mathcal{L}_N = N^2 \{ \mathcal{L}_N^b + \mathcal{L}_N^c + \mathcal{L}_N^{ex} \}, \tag{2.2}$$

where \mathcal{L}_N^b stands for the generator which models the part of the dynamics at which a particle at the boundary can enter or leave the system, \mathcal{L}_N^c stands for the generator which models the collision part of the dynamics and lastly, \mathcal{L}_N^{ex} models the exclusion part of the dynamics. Let f be a function on X_N . The generator of the exclusion part of the dynamics, \mathcal{L}_N^{ex} , is given by

$$(\mathcal{L}_N^{ex} f)(\eta) = \sum_{v \in \mathcal{V}} \sum_{x, z \in D_N^d} \eta(x, v) [1 - \eta(z, v)] P_N(z - x, v) [f(\eta^{x, z, v}) - f(\eta)],$$

where

$$\eta^{x, y, v}(z, w) = \begin{cases} \eta(y, v) & \text{if } w = v \text{ and } z = x, \\ \eta(x, v) & \text{if } w = v \text{ and } z = y, \\ \eta(z, w) & \text{otherwise.} \end{cases}$$

The generator of the collision part of the dynamics, \mathcal{L}_N^c , is given by

$$(\mathcal{L}_N^c f)(\eta) = \sum_{y \in D_N^d} \sum_{q \in \mathcal{Q}} p(y, q, \eta) [f(\eta^{y, q}) - f(\eta)],$$

where \mathcal{Q} is the set of all collisions which preserve momentum:

$$\mathcal{Q} = \{ q = (v, w, v', w') \in \mathcal{V}^4; v + w = v' + w' \},$$

the rate $p(y, q, \eta)$ is given by

$$p(y, q, \eta) = \eta(y, v) \eta(y, w) [1 - \eta(y, v')] [1 - \eta(y, w')],$$

and for $q = (v_0, v_1, v_2, v_3)$, the configuration $\eta^{y, q}$ after the collision is defined as

$$\eta^{y, q}(z, u) = \begin{cases} \eta(y, v_{j+2}) & \text{if } z = y \text{ and } u = v_j \text{ for some } 0 \leq j \leq 3, \\ \eta(z, u) & \text{otherwise,} \end{cases}$$

where the index of v_{j+2} should be taken modulo 4. Particles of velocities v and w at the same site collide at rate one and produce two particles of velocities v' and w' at that site.

Finally, the generator of the boundary part of the dynamics is given by

$$\begin{aligned} (\mathcal{L}_N^b f)(\eta) &= \sum_{\substack{x \in D_N^d \\ x_1=1}} \sum_{v \in \mathcal{V}} [\alpha_v(\tilde{x}/N) [1 - \eta(x, v)] + (1 - \alpha_v(\tilde{x}/N)) \eta(x, v)] [f(\sigma^{x, v} \eta) - f(\eta)] \\ &+ \sum_{\substack{x \in D_N^d \\ x_1=N-1}} \sum_{v \in \mathcal{V}} [\beta_v(\tilde{x}/N) [1 - \eta(x, v)] \\ &+ (1 - \beta_v(\tilde{x}/N)) \eta(x, v)] [f(\sigma^{x, v} \eta) - f(\eta)], \end{aligned}$$

where $\tilde{x} = (x_2, \dots, x_d)$,

$$\sigma^{x,v}\eta(y, w) = \begin{cases} 1 - \eta(x, w), & \text{if } w = v \text{ and } y = x, \\ \eta(y, w), & \text{otherwise} \end{cases}$$

and for every $v \in \mathcal{V}$, $\alpha_v, \beta_v \in C^2(\mathbb{T}^{d-1})$. We also assume that, for every $v \in \mathcal{V}$, α_v and β_v have images belonging to some compact subset of $(0, 1)$. The functions α_v and β_v , which affect the birth and death rates at the two boundaries, represent the densities of the reservoirs.

Note that time has been speeded up diffusively in (2.2). Let $\{\eta(t); t \geq 0\}$ be the Markov process with generator \mathcal{L}_N , and let $D(\mathbb{R}_+, X_N)$ be the set of right continuous functions with left limits taking values on X_N . For a probability measure μ on X_N , denote by \mathbb{P}_μ the measure on the path space $D(\mathbb{R}_+, X_N)$ induced by $\{\eta(t); t \geq 0\}$ and the initial measure μ . Expectation with respect to \mathbb{P}_μ is denoted by \mathbb{E}_μ .

2.2 Mass and Momentum

For each configuration $\xi \in \{0, 1\}^\mathcal{V}$, denote by $I_0(\xi)$ the mass of ξ and by $I_k(\xi)$, $k = 1, \dots, d$, the momentum of ξ :

$$I_0(\xi) = \sum_{v \in \mathcal{V}} \xi(v), \quad I_k(\xi) = \sum_{v \in \mathcal{V}} v_k \xi(v).$$

Set $\mathbf{I}(\xi) := (I_0(\xi), \dots, I_d(\xi))$. Assume that the set of velocities is chosen in such a way that the unique quantities conserved by the random walk dynamics described above are mass and momentum: $\sum_{x \in D_N^d} \mathbf{I}(\eta_x)$. Two examples of sets of velocities satisfying these conditions can be found at [8].

For each chemical potential $\lambda = (\lambda_0, \dots, \lambda_d) \in \mathbb{R}^{d+1}$, denote by m_λ the measure on $\{0, 1\}^\mathcal{V}$ given by

$$m_\lambda(\xi) = \frac{1}{Z(\lambda)} \exp\{\lambda \cdot \mathbf{I}(\xi)\}, \tag{2.3}$$

where $Z(\lambda)$ is a normalizing constant. Note that m_λ is a product measure on $\{0, 1\}^\mathcal{V}$, i.e., that the variables $\{\xi(v); v \in \mathcal{V}\}$ are independent under m_λ .

Denote by μ_λ^N the product measure on X_N , with marginals given by

$$\mu_\lambda^N\{\eta; \eta(x, \cdot) = \xi\} = m_\lambda(\xi),$$

for each ξ in $\{0, 1\}^\mathcal{V}$ and $x \in D_N^d$. Note that $\{\eta(x, v); x \in D_N^d, v \in \mathcal{V}\}$ are independent variables under μ_λ^N , and that the measure μ_λ^N is invariant for the exclusion process with periodic boundary condition.

The expectation under μ_λ^N of the mass and momentum are given by

$$\begin{aligned} \rho(\lambda) &:= E_{\mu_\lambda^N}[I_0(\eta_x)] = \sum_{v \in \mathcal{V}} \theta_v(\lambda), \\ p_k(\lambda) &:= E_{\mu_\lambda^N}[I_k(\eta_x)] = \sum_{v \in \mathcal{V}} v_k \theta_v(\lambda). \end{aligned}$$

In this formula $\theta_v(\lambda)$ denotes the expected value of the density of particles with velocity v under m_λ :

$$\theta_v(\lambda) := E_{m_\lambda}[\xi(v)] = \frac{\exp\{\lambda_0 + \sum_{k=1}^d \lambda_k v_k\}}{1 + \exp\{\lambda_0 + \sum_{k=1}^d \lambda_k v_k\}}.$$

Denote by $(\rho, \mathbf{p})(\lambda) := (\rho(\lambda), p_1(\lambda), \dots, p_d(\lambda))$ the map that associates the chemical potential to the vector of density and momentum. It is possible to prove that (ρ, \mathbf{p}) is a diffeomorphism onto $\mathfrak{U} \subset \mathbb{R}^{d+1}$, the interior of the convex envelope of $\{\mathbf{I}(\xi); \xi \in \{0, 1\}^{\mathcal{V}}\}$. Denote by $\Lambda = (\Lambda_0, \dots, \Lambda_d) : \mathfrak{U} \rightarrow \mathbb{R}^{d+1}$ the inverse of (ρ, \mathbf{p}) . This correspondence allows one to parameterize the invariant states by the density and momentum: for each (ρ, \mathbf{p}) in \mathfrak{U} we have a product measure $\nu_{\rho, \mathbf{p}}^N = \mu_{\Lambda(\rho, \mathbf{p})}^N$ on X_N .

2.3 Dynamical Large Deviations

Fix $T > 0$, let \mathcal{M}_+ be the space of finite positive measures on D^d endowed with the weak topology, and let \mathcal{M} be the space of bounded variation signed measures on D^d endowed with the weak topology. Let $\mathcal{M}_+ \times \mathcal{M}^d$ be the cartesian product of these spaces endowed with the product topology, which is metrizable. Let also \mathcal{M}^0 be the subset of $\mathcal{M}_+ \times \mathcal{M}^d$ of all absolutely continuous measures with respect to the Lebesgue measure satisfying:

$$\mathcal{M}^0 = \{\pi \in \mathcal{M}_+ \times \mathcal{M}^d; \pi(du) = (\rho, \mathbf{p})(u)du, \text{ and } (\rho, \mathbf{p}) \in \mathfrak{U}, \text{ a.e.}\}.$$

Note that if $(\rho, \mathbf{p}) \in \mathfrak{U}$, then $0 \leq \rho(u) \leq |\mathcal{V}|, |p_k(u)| \leq \check{v}|\mathcal{V}|, k = 1, \dots, d$, where $\check{v} = \max_{v \in \mathcal{V}} v_1$. Let $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$ be the set of right continuous functions with left limits taking values on $\mathcal{M}_+ \times \mathcal{M}^d$ endowed with the Skorohod topology. \mathcal{M}^0 is a closed subset of $\mathcal{M}_+ \times \mathcal{M}^d$ and $D([0, T], \mathcal{M}^0)$ is a closed subset of $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$. For a measure $\pi \in \mathcal{M}$, denote by $\langle \pi, G \rangle$ the integral of a function G with respect to π .

Let $\Omega_T = (0, T) \times D^d$ and $\overline{\Omega}_T = [0, T] \times \overline{D^d}$. For $1 \leq m, n \leq +\infty$, denote by $C^{m,n}(\overline{\Omega}_T)$ the space of functions $G = G_t(u) : \overline{\Omega}_T \rightarrow \mathbb{R}$ with m continuous derivatives in time and n continuous derivatives in space. We also denote by $C_0^{m,n}(\overline{\Omega}_T)$ (resp. $C_c^\infty(\Omega_T)$) the set of functions in $C^{m,n}(\overline{\Omega}_T)$ (resp. $C^{\infty,\infty}(\overline{\Omega}_T)$) which vanish at $[0, T] \times \Gamma$ (resp. with compact support in Ω_T).

Let the energy $\mathcal{Q} : D([0, T], \mathcal{M}^0) \rightarrow [0, \infty]$ be given by

$$\mathcal{Q}(\pi) = \sum_{k=0}^d \sum_{i=1}^d \sup_{G \in C_c^\infty(\Omega_T)} \left\{ 2 \int_0^T dt \langle p_{k,t}, \partial_{u_i} G_t \rangle - \int_0^T dt \int_{D^d} G(t, u)^2 du \right\}.$$

where $p_{k,t}(u) = p_k(t, u)$ and $p_{0,t}(u) = \rho(t, u)$.

Let $\mathfrak{C}_0^{1,2}(\overline{\Omega}_T)$ be the set of vector valued function $G = (G^0, \dots, G^d) : [0, T] \times D^d \rightarrow \mathbb{R}^{d+1}$, with each coordinate G_k in $C_0^{1,2}(\overline{\Omega}_T)$, $k = 0, \dots, d$. For each $G \in \mathfrak{C}_0^{1,2}(\overline{\Omega}_T)$ and each measurable function $\gamma = (\rho_0, \mathbf{p}_0) : \overline{D^d} \rightarrow \mathfrak{U}$, let $\hat{J}_G = \hat{J}_{G,\gamma,T} : D([0, T], \mathcal{M}^0) \rightarrow \mathbb{R}$ be the functional given by

$$\begin{aligned} \hat{J}_G(\pi) &= \int_{D^d} G(T, u) \cdot (\rho, \mathbf{p})(T, u) du - \int_{D^d} G(0, u) \cdot (\rho_0, \mathbf{p}_0)(u) du \\ &\quad - \int_0^T dt \int_{D^d} du \left\{ (\rho, \mathbf{p})(t, u) \cdot \partial_t G(t, u) + \frac{1}{2} (\rho, \mathbf{p})(t, u) \cdot \sum_{i=1}^d \partial_{u_i}^2 G(t, u) \right\} \\ &\quad + \frac{1}{2} \int_0^T dt \int_{\{1\} \times \mathbb{T}^{d-1}} dSb(\tilde{u}) \cdot \partial_{u_1} G(t, u) - \frac{1}{2} \int_0^T dt \int_{\{0\} \times \mathbb{T}^{d-1}} dSa(\tilde{u}) \cdot \partial_{u_1} G(t, u) \\ &\quad + \int_0^T dt \int_{D^d} du \sum_{v \in \mathcal{V}} \tilde{v} \cdot \chi_v(\rho, \mathbf{p}) \sum_{i=1}^d v_i \partial_{u_i} G(t, u) \\ &\quad - \int_0^T dt \int_{D^d} du \sum_{v \in \mathcal{V}} \left(\sum_{k=0}^d v_k \partial_{u_i} G_t^k(u) \right)^2 \chi_v(\rho, \mathbf{p}), \end{aligned}$$

where $\chi(r) = r(1 - r)$ is the static compressibility, $\chi_v(\cdot) = \chi(\theta_v(\Lambda(\cdot)))$, for $u = (u_1, \dots, u_d) \in \mathbb{R}^d$, $\tilde{u} = (u_2, \dots, u_d)$, $\pi_t(du) = (\rho, \mathbf{p})(t, u)du$, and dS is the Lebesgue measure on \mathbb{T}^{d-1} . Define $J_G = J_{G,\gamma,T} : D([0, T], \mathcal{M}_+ \times \mathcal{M}^d) \rightarrow \mathbb{R}$ by

$$J_G(\pi) = \begin{cases} \hat{J}_G(\pi), & \text{if } \pi \in D([0, T], \mathcal{M}^0), \\ +\infty, & \text{otherwise.} \end{cases}$$

We define the rate functional $I_T(\cdot|\gamma) : D([0, T], \mathcal{M}_+ \times \mathcal{M}^d) \rightarrow [0, +\infty]$ as

$$I_T(\pi|\gamma) = \begin{cases} \sup_{G \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})} \{J_G(\pi)\}, & \text{if } \mathcal{Q}(\pi) < \infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

We now present the main result of this article, whose proof is given in Sect. 6, which is the dynamical large deviations for this boundary driven exclusion process with many conserved quantities.

Theorem 2.1 Fix $T > 0$ and a measurable function $\gamma = (\rho_0, \mathbf{p}_0) : D^d \rightarrow \mathfrak{A}$. Consider a sequence η^N of configurations in X_N associated to γ in the sense that:

$$\lim_{N \rightarrow \infty} \langle \pi_0^N(\eta^N), G \rangle = \int_{D^d} G(u)\rho_0(u) du,$$

and

$$\lim_{N \rightarrow \infty} \langle \pi_k^N(\eta^N), G \rangle = \int_{D^d} G(u)p_k(u) du, \quad k = 1, \dots, d,$$

for every continuous function $G : \overline{D^d} \rightarrow \mathbb{R}$. Then, the measure $\mathcal{Q}_{\eta^N} = \mathbb{P}_{\eta^N}(\pi^N)^{-1}$ on $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$ satisfies a large deviation principle with speed N^d and rate function $I_T(\cdot|\gamma)$. Namely, for each closed set $\mathcal{C} \subset D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$,

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \log \mathcal{Q}_{\eta^N}(\mathcal{C}) \leq - \inf_{\pi \in \mathcal{C}} I_T(\pi|\gamma)$$

and for each open set $\mathcal{O} \subset D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$,

$$\underline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \log \mathcal{Q}_{\eta^N}(\mathcal{O}) \geq - \inf_{\pi \in \mathcal{O}} I_T(\pi|\gamma).$$

Moreover, the rate function $I_T(\cdot|\gamma)$ is lower semicontinuous and has compact level sets.

3 Hydrodynamics

Fix $T > 0$ and let $(B, \|\cdot\|_B)$ be a Banach space. We denote by $L^2([0, T], B)$ the Banach space of measurable functions $U : [0, T] \rightarrow B$ for which

$$\|U\|_{L^2([0, T], B)}^2 = \int_0^T \|U_t\|_B^2 dt < \infty.$$

Moreover, we denote by $H^1(D^d)$ the Sobolev space of measurable functions in $L^2(D^d)$ that have generalized derivatives in $L^2(D^d)$.

For $x = (x_1, \tilde{x}) \in \{0, 1\} \times \mathbb{T}^{d-1}$, let

$$d(x) = \begin{cases} a(\tilde{x}) = \sum_{v \in \mathcal{V}} (\alpha_v(\tilde{x}), v_1 \alpha_v(\tilde{x}), \dots, v_d \alpha_v(\tilde{x})), & \text{if } x_1 = 0, \\ b(\tilde{x}) = \sum_{v \in \mathcal{V}} (\beta_v(\tilde{x}), v_1 \beta_v(\tilde{x}), \dots, v_d \beta_v(\tilde{x})), & \text{if } x_1 = 1. \end{cases} \tag{3.1}$$

Fix a bounded density profile $\rho_0 : D^d \rightarrow \mathbb{R}_+$, and a bounded momentum profile $\mathbf{p}_0 : D^d \rightarrow \mathbb{R}^d$. A bounded function $(\rho, \mathbf{p}) : [0, T] \times D^d \rightarrow \mathbb{R}_+ \times \mathbb{R}^d$ is a weak solution of the system of parabolic partial differential equations

$$\begin{cases} \partial_t(\rho, \mathbf{p}) + \sum_{v \in \mathcal{V}} \tilde{v} [v \cdot \nabla \chi_v(\rho, \mathbf{p})] = \frac{1}{2} \Delta(\rho, \mathbf{p}), \\ (\rho, \mathbf{p})(0, \cdot) = (\rho_0, \mathbf{p}_0)(\cdot) \quad \text{and} \quad (\rho, \mathbf{p})(t, x) = d(x), \quad x \in \{0, 1\} \times \mathbb{T}^{d-1}, \end{cases} \tag{3.2}$$

if for every vector valued function $H \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})$, we have

$$\begin{aligned} & \int_{D^d} H(T, u) \cdot (\rho, \mathbf{p})(T, u) du - \int_{D^d} H(0, u) \cdot (\rho_0, \mathbf{p}_0)(u) du \\ &= \int_0^T dt \int_{D^d} du \left\{ (\rho, \mathbf{p})(t, u) \cdot \partial_t H(t, u) + \frac{1}{2} (\rho, \mathbf{p})(t, u) \cdot \sum_{i=1}^d \partial_{u_i}^2 H(t, u) \right\} \\ & \quad - \frac{1}{2} \int_0^T dt \int_{\{1\} \times \mathbb{T}^{d-1}} dS b(\tilde{u}) \cdot \partial_{u_1} H(t, u) + \frac{1}{2} \int_0^T dt \int_{\{0\} \times \mathbb{T}^{d-1}} dS a(\tilde{u}) \cdot \partial_{u_1} H(t, u) \\ & \quad - \int_0^T dt \int_{D^d} du \sum_{v \in \mathcal{V}} \tilde{v} \cdot \chi_v(\rho, \mathbf{p}) \sum_{i=1}^d v_i \partial_{u_i} H(t, u). \end{aligned}$$

We say that the solution (ρ, \mathbf{p}) has finite energy if its components belong to $L^2([0, T], H^1(D^d))$:

$$\int_0^T ds \left(\int_{D^d} \|\nabla \rho(s, u)\|^2 du \right) < \infty,$$

and

$$\int_0^T ds \left(\int_{D^d} \|\nabla p_k(s, u)\|^2 du \right) < \infty,$$

for $k = 1, \dots, d$, where ∇f represents the generalized gradient of the function f .

In [12] the following theorem was proved:

Theorem 3.1 *Let $(\mu^N)_N$ be a sequence of probability measures on X_N associated to the profile (ρ_0, \mathbf{p}_0) in the sense of Theorem 2.1. Then, for every $t \geq 0$, for every continuous function $H : D^d \rightarrow \mathbb{R}$ vanishing at the boundary Γ , and for every $\delta > 0$,*

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\mu^N} \left[\left| \frac{1}{N^d} \sum_{x \in D_N^d} H\left(\frac{x}{N}\right) I_0(\eta_x(t)) - \int_{D^d} H(u) \rho(t, u) du \right| > \delta \right] = 0,$$

and for $1 \leq k \leq d$

$$\lim_{N \rightarrow \infty} \mathbb{P}^{\mu^N} \left[\left| \frac{1}{N^d} \sum_{x \in D_N^d} H\left(\frac{x}{N}\right) I_k(\eta_x(t)) - \int_{D^d} H(u) p_k(t, u) du \right| > \delta \right] = 0,$$

where (ρ, \mathbf{p}) has finite energy and is the unique weak solution of (3.2).

4 The Rate Function $I_T(\cdot|\gamma)$

We examine in this section the rate function $I_T(\cdot|\gamma)$. The main result, presented in Theorem 4.6 below, states that $I_T(\cdot|\gamma)$ has compact level sets. The proof relies on two ingredients. The first one, stated in Lemma 4.2, is an estimate of the energy and of the H_{-1} norm of the time derivative of a trajectory in terms of the rate function. The second one, stated in Lemma 4.5, establishes that sequences of trajectories, with rate function uniformly bounded, which converge weakly in L^2 converge in fact strongly. We follow the strategy introduced in [9].

Let V be an open neighborhood of D^d , and consider, for each $v \in \mathcal{V}$, smooth functions $\kappa_k^v : V \rightarrow (0, 1)$ in $C^2(V)$, for $k = 0, \dots, d$. We assume that the restriction of $\kappa = \sum_{v \in \mathcal{V}} (\kappa_0^v, v_1 \kappa_1^v, \dots, v_d \kappa_d^v)$ to $\{0\} \times \mathbb{T}^{d-1}$ equals the vector valued function $a(\cdot)$ defined in (3.1), and that the restriction of κ to $\{1\} \times \mathbb{T}^{d-1}$ equals the vector valued function $b(\cdot)$, also defined in (3.1), in the sense that $\kappa(x) = d(x_1, \tilde{x})$ if $x \in \{0, 1\} \times \mathbb{T}^{d-1}$.

Let $L^2(D^d)$ be the Hilbert space of functions $G : D^d \rightarrow \mathbb{R}$ such that $\int_{D^d} |G(u)|^2 du < \infty$ equipped with the inner product

$$\langle G, F \rangle_2 = \int_{\Omega} G(u) F(u) du,$$

and the norm of $L^2(D^d)$ is denoted by $\|\cdot\|_2$.

Recall that $H^1(D^d)$ is the Sobolev space of functions G with generalized derivatives $\partial_{u_1} G, \dots, \partial_{u_d} G$ in $L^2(D^d)$. $H^1(D^d)$ endowed with the scalar product $\langle \cdot, \cdot \rangle_{1,2}$, defined by

$$\langle G, F \rangle_{1,2} = \langle G, F \rangle_2 + \sum_{j=1}^d \langle \partial_{u_j} G, \partial_{u_j} F \rangle_2,$$

is a Hilbert space. The corresponding norm is denoted by $\|\cdot\|_{1,2}$.

Recall also that we denote by $C_c^\infty(D^d)$ the set of infinitely differentiable functions $G : D^d \rightarrow \mathbb{R}$, with compact support in D^d . Denote by $H_0^1(D^d)$ the closure of $C_c^\infty(D^d)$ in $H^1(D^d)$. Since D^d is bounded, by Poincaré’s inequality, there exists a finite constant C such that for all $G \in H_0^1(D^d)$

$$\|G\|_2^2 \leq C \sum_{j=1}^d \langle \partial_{u_j} G, \partial_{u_j} G \rangle_2.$$

This implies that, in $H_0^1(D^d)$

$$\|G\|_{1,2,0} = \left\{ \sum_{j=1}^d \langle \partial_{u_j} G, \partial_{u_j} G \rangle_2 \right\}^{1/2}$$

is a norm equivalent to the norm $\| \cdot \|_{1,2}$. Moreover, $H_0^1(D^d)$ is a Hilbert space with inner product given by

$$\langle G, J \rangle_{1,2,0} = \sum_{j=1}^d \langle \partial_{u_j} G, \partial_{u_j} J \rangle_2.$$

To assign boundary values along the boundary Γ of D^d to any function G in $H^1(D^d)$, recall, from the trace Theorem [15, Theorem 21.A.(e)], that there exists a continuous linear operator $\text{Tr} : H^1(D^d) \rightarrow L^2(\Gamma)$, called trace, such that $\text{Tr}(G) = G|_\Gamma$ if $G \in H^1(D^d) \cap C(\overline{D^d})$. Moreover, the space $H_0^1(D^d)$ is the space of functions G in $H^1(D^d)$ with zero trace [15, Appendix (48b)]:

$$H_0^1(D^d) = \{G \in H^1(D^d); \text{Tr}(G) = 0\}.$$

Finally, denote by $H^{-1}(D^d)$ the dual of $H_0^1(D^d)$. $H^{-1}(D^d)$ is a Banach space with norm $\| \cdot \|_{-1}$ given by

$$\|v\|_{-1}^2 = \sup_{G \in C_c^\infty(D^d)} \left\{ 2\langle v, G \rangle_{-1,1} - \int_{D^d} \|\nabla G(u)\|^2 du \right\},$$

where $\langle v, G \rangle_{-1,1}$ stands for the values of the linear form v at G .

For each $G \in C_c^\infty(\Omega_T)$ and each integer $1 \leq i \leq d$, let $\mathcal{Q}_{i,k}^G : D([0, T], \mathcal{M}^0) \rightarrow \mathbb{R}$ be the functional given by

$$\mathcal{Q}_{i,k}^G(\pi) = 2 \int_0^T dt \langle \pi_t^k, \partial_{u_i} G_t \rangle - \int_0^T dt \int_{D^d} du G(t, u)^2,$$

where $\pi = (\pi^0, \pi^1, \dots, \pi^d)$. Recall, from Sect. 2.2, that the energy $\mathcal{Q}(\pi)$ is given by

$$\mathcal{Q}(\pi) = \sum_{k=0}^d \sum_{i=1}^d \mathcal{Q}_{i,k}(\pi), \quad \text{with} \quad \mathcal{Q}_{i,k}(\pi) = \sup_{G \in C_c^\infty(\Omega_T)} \mathcal{Q}_{i,k}^G(\pi).$$

The functional $\mathcal{Q}_{i,k}^G$ is convex and continuous in the Skorohod topology. Therefore $\mathcal{Q}_{i,k}$ and \mathcal{Q} are convex and lower semicontinuous. Furthermore, it is well known that a measure $\pi(t, du) = (\rho, \mathbf{p})(t, u)du$ in $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$ has finite energy, $\mathcal{Q}(\pi) < \infty$, if and only if its density ρ and its momentum \mathbf{p} belong to $L^2([0, T], H^1(D^d))$. In which case

$$\mathcal{Q}(\pi) := \sum_{k=0}^d \int_0^T dt \int_{D^d} du \|\nabla p_{k,t}(u)\|^2 < \infty,$$

where $p_{0,t}(u) = \rho(t, u)$.

Let $D_\gamma = D_{\gamma,d}$ be the subset of $C([0, T], \mathcal{M}^0)$ consisting of all paths $\pi(t, du) = (\rho, \mathbf{p})(t, u)du$ with initial profile $\gamma(\cdot) = (\rho_0, \mathbf{p}_0)(\cdot)$, finite energy $\mathcal{Q}(\pi)$ (in which case ρ_t and \mathbf{p}_t belong to $H^1(D^d)$ for almost all $0 \leq t \leq T$ and so $\text{Tr}(\rho_t)$ is well defined for those t) and such that $\text{Tr}(\rho_t) = d_0$ and $\text{Tr}(p_{k,t}) = d_k, k = 1, \dots, d$, for almost all t in $[0, T]$, where $d(\cdot) = (d_0(\cdot), d_1(\cdot), \dots, d_d(\cdot))$.

Lemma 4.1 *Let π be a trajectory in $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$ such that $I_T(\pi|\gamma) < \infty$. Then π belongs to D_γ .*

Proof Fix a path π in $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$ with finite rate function, $I_T(\pi|\gamma) < \infty$. By definition of I_T , π belongs to $D([0, T], \mathcal{M}^0)$. Denote its density and momentum by (ρ, \mathbf{p}) : $\pi(t, du) = (\rho, \mathbf{p})(t, u)du$.

The proof that $(\rho, \mathbf{p})(0, \cdot) = \gamma(\cdot)$ is similar to the one of Lemma 3.5 in [6], and the proof that $\text{Tr}(\rho_t) = d_0, \text{Tr}(p_{k,t}) = d_k, k = 1, \dots, d$, is similar to the one found in Lemma 4.1 in [9].

We deal now with the continuity of π . We claim that there exists a positive constant C_0 such that, for any $g \in [C_c^\infty(D^d)]^{d+1}$, and any $0 \leq s < r < T$,

$$|\langle \pi_r, g \rangle - \langle \pi_s, g \rangle| \leq C_0(r - s)^{1/2} \{C_1 + I_T(\pi|\gamma) + \|g\|_{1,2,0}^2 + (r - s)^{1/2} \|\Delta g\|_1\}. \tag{4.1}$$

Indeed, for each $\delta > 0$, let $\psi^\delta : [0, T] \rightarrow \mathbb{R}$ be the function given by

$$(r - s)^{1/2} \psi^\delta(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq s \text{ or } r + \delta \leq t \leq T, \\ \frac{t-s}{\delta} & \text{if } s \leq t \leq s + \delta, \\ 1 & \text{if } s + \delta \leq t \leq r, \\ 1 - \frac{t-r}{\delta} & \text{if } r \leq t \leq r + \delta, \end{cases}$$

and let $G_\epsilon^\delta(t, u) = \psi_\epsilon^\delta(t)g(u)$, where $\psi_\epsilon^\delta(\cdot)$ is the standard ϵ -mollification of $\psi^\delta(\cdot)$. Since G_ϵ^δ is in $\mathfrak{C}_0^{1,2}(\overline{\Omega_T})$, we have

$$\begin{aligned} (r - s)^{1/2} \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} J_{G_\epsilon^\delta}(\pi) &= \langle \pi_r, g \rangle - \langle \pi_s, g \rangle - \int_s^r dt \langle \pi_t, \Delta g \rangle \\ &+ \int_r^s dt \int_{D^d} du \sum_{v \in \mathcal{V}} \tilde{v} \cdot \chi_v(\rho, \mathbf{p}) \sum_{i=1}^d v_i \partial_{u_i} g(u) \\ &- \frac{1}{(r - s)^{1/2}} \int_s^r dt \int_{D^d} du \sum_{v \in \mathcal{V}} \left(\sum_{k=0}^d v_k \partial_{u_i} g^k(u) \right)^2 \chi_v(\rho, \mathbf{p}). \end{aligned}$$

To conclude the proof, we observe that the left-hand side is bounded by $(r - s)^{1/2} I_T(\pi|\gamma)$, that χ is positive and bounded above on $[0, 1]$ by $1/4$, and finally, we use the elementary inequality $2ab \leq a^2 + b^2$. □

Denote by $L^2([0, T], H_0^1(D^d))^*$ the dual of $L^2([0, T], H_0^1(D^d))$. By Proposition 23.7 in [15], $L^2([0, T], H_0^1(D^d))^*$ corresponds to $L^2([0, T], H^{-1}(D^d))$ and for v in $L^2([0, T], H_0^1(D^d))^*$, G in $L^2([0, T], H_0^1(D^d))$,

$$\langle\langle v, G \rangle\rangle_{-1,1} = \int_0^T \langle v_t, G_t \rangle_{-1,1} dt, \tag{4.2}$$

where the left hand side stands for the value of the linear functional v at G . Moreover, if we denote by $\|v\|_{-1}$ the norm of v ,

$$\|v\|_{-1}^2 = \int_0^T \|v_t\|_{-1}^2 dt.$$

Fix a path $\pi(t, du) = (\rho, \mathbf{p})(t, u)du$ in D_γ and suppose that for $k = 0, \dots, d$

$$\sup_{G \in \mathfrak{C}_c^\infty(\Omega_T)} \left\{ 2 \int_0^T dt \langle p_{k,t}, \partial_t G_t \rangle_2 - \int_0^T dt \int_{D^d} du \|\nabla G_t\|^2 \right\} < \infty. \tag{4.3}$$

In this case, for each k , $\partial_t p_k : C_c^\infty(\Omega_T) \rightarrow \mathbb{R}$ defined by

$$\partial_t p_k(G) = - \int_0^T \langle p_{k,t}, \partial_t G_t \rangle_2 dt$$

can be extended to a bounded linear operator $\partial_t p_k : L^2([0, T], H_0^1(D^d)) \rightarrow \mathbb{R}$. It belongs therefore to $L^2([0, T], H_0^1(D^d))^* = L^2([0, T], H^{-1}(D^d))$. In particular, there exists $v^k = \{v_t^k; 0 \leq t \leq T\}$ in $L^2([0, T], H^{-1}(D^d))$, which we denote by $v_t^k = \partial_t p_{k,t}$, such that for any G in $L^2([0, T], H_0^1(D^d))$,

$$\langle \partial_t p_k, G \rangle_{-1,1} = \int_0^T \langle \partial_t p_{k,t}, G_t \rangle_{-1,1} dt.$$

Moreover,

$$\begin{aligned} \|\partial_t p_k\|_{-1}^2 &= \int_0^T \|\partial_t p_{k,t}\|_{-1}^2 dt \\ &= \sup_{G \in C_c^\infty(\Omega_T)} \left\{ 2 \int_0^T dt \langle p_{k,t}, \partial_t G_t \rangle_2 - \int_0^T dt \int_{D^d} du \|\nabla G_t\|^2 \right\}. \end{aligned}$$

Denote by $\langle \partial_t(\rho, \mathbf{p}), \cdot \rangle_{-1,1} : L^2([0, T], [H_0^1(D^d)]^{d+1}) \rightarrow \mathbb{R}$ the linear functional given by

$$\langle \partial_t(\rho, \mathbf{p}), G \rangle_{-1,1} = \sum_{k=0}^d \langle \partial_t p_k, G^k \rangle_{-1,1},$$

with $G = (G^0, \dots, G^d)$, and

$$\|\partial_t(\rho, \mathbf{p})\|_{-1}^2 = \sum_{k=0}^d \|\partial_t p_k\|_{-1}^2.$$

Let W be the set of paths $\pi(t, du) = (\rho, \mathbf{p})(t, u)du$ in D_γ such that (4.3) holds, i.e., such that $\partial_t p_k$ belongs to $L^2([0, T], H^{-1}(D^d))$. For G in $L^2([0, T], [H_0^1(D^d)]^{d+1})$, let $\mathbb{J}_G : W \rightarrow \mathbb{R}$ be the functional given by

$$\begin{aligned} \mathbb{J}_G(\pi) &= \langle \partial_t(\rho, \mathbf{p}), G \rangle_{-1,1} + \frac{1}{2} \int_0^T dt \int_{D^d} du \sum_{i=1}^d \partial_{u_i}(\rho, \mathbf{p})(t, u) \cdot \partial_{u_i} G(t, u) \\ &\quad + \int_0^T dt \int_{D^d} du \sum_{v \in \mathcal{V}} \tilde{v} \cdot \chi_v(\rho, \mathbf{p}) \sum_{i=1}^d v_i \partial_{u_i} G(t, u) \\ &\quad - \int_0^T dt \int_{D^d} du \sum_{v \in \mathcal{V}} (\tilde{v} \cdot \partial_{u_i} G_t(u))^2 \chi_v(\rho, \mathbf{p}), \end{aligned}$$

Note that $\mathbb{J}_G(\pi) = J_G(\pi)$ for every G in $C_c^\infty(\Omega_T) \times [C_c^\infty(D^d)]^d$. Moreover, since $\mathbb{J}(\pi)$ is continuous in $L^2([0, T], [H_0^1(D^d)]^{d+1})$ and since $C_c^\infty(\Omega_T)$ is dense in $C_0^{1,2}(\overline{\Omega_T})$ and in $L^2([0, T], H_0^1(D^d))$, for every π in W ,

$$I_T(\pi|\gamma) = \sup_{G \in C_c^\infty(\Omega_T) \times [C_c^\infty(D^d)]^d} \mathbb{J}_G(\pi) = \sup_{G \in L^2([0, T], [H_0^1(D^d)]^{d+1})} \mathbb{J}_G(\pi). \tag{4.4}$$

Lemma 4.2 *There exists a constant $C_0 > 0$ such that if the density and momentum (ρ, \mathbf{p}) of some path $\pi(t, du) = (\rho, \mathbf{p})(t, u)du$ in $D([0, T], \mathcal{M}^0)$ has generalized gradients, $\nabla \rho$ and $\nabla p_k, k = 1, \dots, d$. Then*

$$\|\partial_t(\rho, \mathbf{p})\|_{-1}^2 \leq C_0\{I_T(\pi|\gamma) + \mathcal{Q}(\pi)\}, \tag{4.5}$$

$$\sum_{k=0}^d \int_0^T dt \int_{D^d} du \|\nabla p_k(t, u)\|^2 \leq C_0\{I_T(\pi|\gamma) + 1\}, \tag{4.6}$$

where $p_0 = \rho$.

Proof Fix a path $\pi(t, du) = (\rho, \mathbf{p})(t, u)du$ in $D([0, T], \mathcal{M}^0)$. In view of the discussion presented before the lemma, we need to show that the left hand side of (4.3) is bounded by the right hand side of (4.5). Such an estimate follows from the definition of the rate function $I_T(\cdot|\gamma)$ and from the elementary inequality $2ab \leq Aa^2 + A^{-1}b^2$.

To prove (4.6), observe that

$$\begin{aligned} I_T(\pi|\gamma) &\geq J_G(\pi) = \partial_t \pi(G) + \frac{1}{2} \int_0^T dt \int_{D^d} du \sum_{i=1}^d \partial_{u_i}(\rho, p) \cdot \partial_{u_i} G \\ &\quad + \int_0^T dt \int_{D^d} du \sum_{v \in \mathcal{V}} \chi_v(\rho, p) \sum_{i=1}^d \tilde{v} \cdot (v_i \partial_{u_i} G) \\ &\quad - \int_0^T dt \int_{D^d} du \sum_{v \in \mathcal{V}} \sum_{i=1}^d (\tilde{v} \cdot \partial_{u_i} G)^2 \chi_v(\rho, p) \\ &\geq \partial_t \pi(G) + \frac{1}{2} \int_0^T dt \int_{D^d} du \sum_{i=1}^d \partial_{u_i}(\rho, p) \cdot \partial_{u_i} G - C \int_0^T dt \int_{D^d} du \sum_{k=0}^d \|\nabla G^k\|^2, \end{aligned}$$

where C is constant obtained from the elementary inequality $2ab \leq a^2 + b^2$, the fact that \mathcal{V} is finite, and that χ is bounded above by $1/4$ in $[0, 1]$.

Recall the definition of the function κ given at the beginning of Sect. 4. Now, consider $G = K(\pi - \kappa)$, $K > 0$ being a constant, and note that $\pi - \kappa$ belongs to $L^2([0, T], H_0^1(D^d))$, which implies that it may be approximated by C_c^∞ functions. Therefore $|\partial_t \pi(G)| = K|\langle \pi_T, \pi_T/2 - \kappa \rangle - \langle \pi_0, \pi_0/2 - \kappa \rangle|$, which is bounded from above by some constant C_1 . We, then, obtain that

$$\begin{aligned} I(\pi) &\geq \int_0^T dt \int_{D^d} du \left\{ -C_1 + \frac{K}{2} \sum_{k=0}^d \|\nabla p_k\|^2 - \frac{K}{2} \sum_{i=1}^d \partial_{u_i}(\rho, p) \cdot \partial_{u_i} \kappa \right. \\ &\quad \left. - CK^2 \sum_{k=0}^d \|\nabla(p_k - \kappa_k)\|^2 \right\} \\ &\geq \int_0^T dt \int_{D^d} du \left\{ (K/4 - 2CK^2) \sum_{k=0}^d \|\nabla p_k\|^2 - \frac{K}{4} \sum_{k=0}^d \|\nabla \kappa_k\|^2 \right. \\ &\quad \left. - 2CK^2 \sum_{k=0}^d \|\nabla \kappa_k\|^2 - C_1 \right\}, \end{aligned}$$

where in the last inequality we used the Cauchy-Schwartz inequality and the elementary inequality $2ab \leq a^2 + b^2$. The proof thus follows from choosing a suitable K , the estimate given in (4.5), and the fact we have a fixed smooth function κ . \square

Corollary 4.3 *The density (ρ, \mathbf{p}) of a path $\pi(t, du) = (\rho, \mathbf{p})(t, u)du$ in $D([0, T], \mathcal{M}^0)$ is the weak solution of (3.2) and initial profile γ if and only if the rate function $I_T(\pi|\gamma)$ vanishes. Moreover, if any of the above conditions hold, π has finite energy ($\mathcal{Q}(\pi) < \infty$).*

Proof On the one hand, if the density (ρ, \mathbf{p}) of a path $\pi(t, du) = (\rho, \mathbf{p})(t, u)du$ in $D([0, T], \mathcal{M}^0)$ is the weak solution of (3.2) with initial condition is γ , in the formula of $\hat{J}_G(\pi)$, the linear part in G vanishes which proves that the rate functional $I_T(\pi|\gamma)$ vanishes. On the other hand, if the rate functional vanishes, the path (ρ, \mathbf{p}) belongs to $L^2([0, T], [H^1(D^d)]^{d+1})$ and the linear part in G of $J_G(\pi)$ has to vanish for all functions G . In particular, (ρ, \mathbf{p}) is a weak solution of (3.2). Moreover, if the rate function is finite, by the previous lemma, π has finite energy. Accordingly, if π is a weak solution, we have from Theorem 3.1 that it has finite energy. \square

For each $q > 0$, let E_q be the level set of $I_T(\pi|\gamma)$ defined by

$$E_q = \{\pi \in D([0, T], \mathcal{M}_+ \times \mathcal{M}^d); I_T(\pi|\gamma) \leq q\}.$$

By Lemma 4.1, E_q is a subset of $C([0, T], \mathcal{M}^0)$. Thus, from the previous lemma, it is easy to deduce the next result.

Corollary 4.4 *For every $q \geq 0$, there exists a finite constant $C(q)$ such that*

$$\sup_{\pi \in E_q} \left\{ \|\partial_t(\rho, \mathbf{p})\|_{-1}^2 + \sum_{k=0}^d \int_0^T dt \int_{D^d} du \|\nabla p_k(t, u)\|^2 \right\} \leq C(q).$$

Next result together with the previous estimates provide the compactness needed in the proof of the lower semicontinuity of the rate function.

Lemma 4.5 *Let $\{\rho^n; n \geq 1\}$ be a sequence of functions in $L^2(\Omega_T)$ such that uniformly on n ,*

$$\int_0^T dt \|\rho_t^n\|_{1,2}^2 + \int_0^T dt \|\partial_t \rho_t^n\|_{-1}^2 \leq C$$

for some positive constant C . Suppose that $\rho \in L^2(\Omega_T)$ and that $\rho^n \rightarrow \rho$ weakly in $L^2(\Omega_T)$. Then $\rho^n \rightarrow \rho$ strongly in $L^2(\Omega_T)$.

Proof Since $H^1(D^d) \subset L^2(D^d) \subset H^{-1}(D^d)$ with compact embedding $H^1(D^d) \rightarrow L^2(D^d)$, from Corollary 8.4, [13], the sequence $\{\rho_n\}$ is relatively compact in $L^2([0, T], L^2(D^d))$. Therefore the weak convergence implies the strong convergence in $L^2([0, T], L^2(D^d))$. \square

Theorem 4.6 *The functional $I_T(\cdot|\gamma)$ is lower semicontinuous and has compact level sets.*

Proof We have to show that, for all $q \geq 0$, E_q is compact in $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$. Since $E_q \subset C([0, T], \mathcal{M}^0)$ and $C([0, T], \mathcal{M}^0)$ is a closed subset of $D([0, T], \mathcal{M})$, we just need to show that E_q is compact in $C([0, T], \mathcal{M}^0)$.

We will show first that E_q is closed in $C([0, T], \mathcal{M}^0)$. Fix $q \in \mathbb{R}$ and let $\{\pi^n; n \geq 1\}$ be a sequence in E_q converging to some π in $C([0, T], \mathcal{M}^0)$. Then, for all $G \in \mathcal{C}(\overline{\Omega_T}) \times [\mathcal{C}(\overline{D^d})]^d$,

$$\lim_{n \rightarrow \infty} \int_0^T dt \langle \pi_t^n, G_t \rangle = \int_0^T dt \langle \pi_t, G_t \rangle.$$

Notice that this means that $\pi^{n,k} \rightarrow \pi^k$ weakly in $L^2(\Omega_T)$, for each $k = 0, \dots, d$, which together with Corollary 4.4 and Lemma 4.5 imply that $\pi^{n,k} \rightarrow \pi^k$ strongly in $L^2(\Omega_T)$. From this fact and the definition of J_G it is easy to see that, for all G in $\mathfrak{C}_0^{1,2}(\overline{\Omega_T})$,

$$\lim_{n \rightarrow \infty} J_G(\pi_n) = J_G(\pi).$$

This limit, Corollary 4.4 and the lower semicontinuity of \mathcal{Q} permit us to conclude that $\mathcal{Q}(\pi) \leq C(q)$ and that $I_T(\pi|\gamma) \leq q$.

We prove now that E_q is relatively compact. To this end, it is enough to prove that for every continuous function $G : \overline{D^d} \rightarrow \mathbb{R}$, and every $k = 0, \dots, d$,

$$\limsup_{\delta \rightarrow 0} \sup_{\pi \in E_q} \sup_{\substack{0 \leq s, r \leq T \\ |r-s| < \delta}} |\langle \pi_r^k, G \rangle - \langle \pi_s^k, G \rangle| = 0. \tag{4.7}$$

Since $E_q \subset C([0, T], \mathcal{M}^0)$, we may assume by approximations of G in $L^1(D^d)$ that $G \in \mathcal{C}_c^\infty(D^d)$. In which case, (4.7) follows from (4.1). □

We conclude this section with an explicit formula for the rate function $I_T(\cdot|\gamma)$. For each $\pi(t, du) = (\rho, \mathbf{p})(t, u)du$ in $D([0, T], \mathcal{M}^0)$, denote by $H_0^1(\pi)$ the Hilbert space induced by $\mathfrak{C}_0^{1,2}(\overline{\Omega_T})$ endowed with the inner product $\langle \cdot, \cdot \rangle_\pi$ defined by

$$\langle H, G \rangle_\pi = \sum_{v \in \mathcal{V}} \sum_{i=1}^d \int_0^T dt \int_{D^d} du \chi_v(\rho, \mathbf{p}) [\tilde{v} \cdot \partial_{u_i} H][\tilde{v} \cdot \partial_{u_i} G]. \tag{4.8}$$

Induced means that we first declare two functions F, G in $\mathfrak{C}_0^{1,2}(\overline{\Omega_T})$ to be equivalent if $\langle F - G, F - G \rangle_\pi = 0$, and then we complete the quotient space with respect to the inner product $\langle \cdot, \cdot \rangle_\pi$. The norm of $H_0^1(\pi)$ is denoted by $\| \cdot \|_\pi$.

Fix a path π in $D([0, T], \mathcal{M}^0)$ and a function H in $H_0^1(\pi)$. A measurable function $\lambda : [0, T] \times D^d \rightarrow \mathbb{R}_+ \times \mathbb{R}^d$ is said to be a weak solution of the nonlinear boundary value parabolic equation

$$\begin{cases} \partial_t \lambda + \sum_{i=1}^d \sum_{v \in \mathcal{V}} \tilde{v} \partial_{u_i} [\chi_v(\lambda)(v_i - \tilde{v} \cdot \partial_{u_i} H)] = \frac{1}{2} \Delta \lambda, \\ \lambda(0, \cdot) = \gamma(\cdot), \\ \lambda(t, x) = d(x), x \in \{0, 1\} \times \mathbb{T}^{d-1}, \end{cases} \tag{4.9}$$

if it satisfies the following two conditions:

- (i) For $k = 0, \dots, d$, λ_k belongs to $L^2([0, T], H^1(D^d))$:

$$\int_0^T ds \left(\int_{D^d} \|\nabla \lambda_k(s, u)\|^2 du \right) < \infty;$$

(ii) For every function $G(t, u) = G_t(u)$ in $\mathfrak{C}_0^{1,2}(\overline{\Omega_T})$,

$$\begin{aligned} & \int_{D^d} G(T, u) \cdot \lambda(T, u) du - \int_{D^d} G(0, u) \cdot \gamma(u) du \\ &= \int_0^T dt \int_{D^d} du \left\{ \lambda(t, u) \cdot \partial_t G(t, u) + \frac{1}{2} \lambda(t, u) \cdot \sum_{i=1}^d \partial_{u_i}^2 G(t, u) \right\} \\ & \quad - \frac{1}{2} \int_0^T dt \int_{\{1\} \times \mathbb{T}^{d-1}} dSb(\tilde{u}) \cdot \partial_{u_1} G(t, u) + \frac{1}{2} \int_0^T dt \int_{\{0\} \times \mathbb{T}^{d-1}} dSa(\tilde{u}) \cdot \partial_{u_1} G(t, u) \\ & \quad - \int_0^T dt \int_{D^d} du \sum_{v \in \mathcal{V}} \tilde{v} \cdot \chi_v(\lambda) \sum_{i=1}^d v_i \partial_{u_i} G(t, u) \\ & \quad + \sum_{v \in \mathcal{V}} \sum_{i=1}^d \int_0^T dt \int_{D^d} du \chi_v(\lambda) [\tilde{v} \cdot \partial_{u_i} H][\tilde{v} \cdot \partial_{u_i} G]. \end{aligned}$$

Uniqueness of solutions of (4.9) follows from the same arguments of the uniqueness proved in [12].

Lemma 4.7 Assume that $\pi(t, du) = (\rho, \mathbf{p})(t, u) du$ in $D([0, T], \mathcal{M}^0)$ has finite rate function: $I_T(\pi|\gamma) < \infty$. Then, there exists a function H in $H_0^1(\pi)$ such that (ρ, \mathbf{p}) is a weak solution to (4.9). Moreover,

$$I_T(\pi|\gamma) = \frac{1}{4} \|H\|_\pi^2. \tag{4.10}$$

The proof of this lemma is similar to the one of Lemma 10.5.3 in [10] and is therefore omitted.

5 $I_T(\cdot|\gamma)$ -Density

The main result of this section, stated in Theorem 5.5, asserts that any trajectory $\lambda_t, 0 \leq t \leq T$, with finite rate function, $I_T(\lambda|\gamma) < \infty$, can be approximated by a sequence of smooth trajectories $\{\lambda^n; n \geq 1\}$ such that

$$\lambda^n \rightarrow \lambda \quad \text{and} \quad I_T(\lambda^n|\gamma) \rightarrow I_T(\lambda|\gamma).$$

This is one of the main steps in the proof of the lower bound of the large deviations principle for the empirical measure. The proof is mainly based on the regularizing effects of the hydrodynamic equation. This strategy was introduced in [9].

A subset A of $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$ is said to be $I_T(\cdot|\gamma)$ -dense if for every π in $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$ such that $I_T(\pi|\gamma) < \infty$, there exists a sequence $\{\pi^n; n \geq 1\}$ in A such that π^n converges to π and $I_T(\pi^n|\gamma)$ converges to $I_T(\pi|\gamma)$.

Let Π_1 be the subset of $D([0, T], \mathcal{M}^0)$ consisting of paths $\pi(t, du) = (\rho, \mathbf{p})(t, u) du$ whose density (ρ, \mathbf{p}) is a weak solution of the hydrodynamic equation (3.2) in the time interval $[0, \delta]$ for some $\delta > 0$.

Lemma 5.1 The set Π_1 is $I_T(\cdot|\gamma)$ -dense.

Proof Fix $\pi(t, du) = (\rho, \mathbf{p})(t, u)du$ in $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$ such that $I_T(\pi|\gamma) < \infty$. By Lemma 4.1, π belongs to $C([0, T], \mathcal{M}^0)$. For each $\delta > 0$, let $(\rho^\delta, \mathbf{p}^\delta)$ be the path defined as

$$(\rho^\delta, \mathbf{p}^\delta)(t, u) = \begin{cases} \tau(t, u) & \text{if } 0 \leq t \leq \delta, \\ \tau(2\delta - t, u) & \text{if } \delta \leq t \leq 2\delta, \\ (\rho, \mathbf{p})(t - 2\delta, u) & \text{if } 2\delta \leq t \leq T, \end{cases}$$

where τ is the weak solution of the hydrodynamic equation (3.2) starting at γ . It is clear that $\pi^\delta(t, du) = (\rho^\delta, \mathbf{p}^\delta)(t, u)du$ belongs to D_γ , because so do π and τ and that $\mathcal{Q}(\pi^\delta) \leq \mathcal{Q}(\pi) + 2\mathcal{Q}(\tau) < \infty$. Moreover, π^δ converges to π as $\delta \downarrow 0$ because π belongs to $C([0, T], \mathcal{M}^0)$. By the lower semicontinuity of $I_T(\cdot|\gamma)$, $I_T(\pi|\gamma) \leq \liminf_{\delta \rightarrow 0} I_T(\pi^\delta|\gamma)$. Then, in order to prove the lemma, it is enough to prove that $I_T(\pi|\gamma) \geq \limsup_{\delta \rightarrow 0} I_T(\pi^\delta|\gamma)$. To this end, decompose the rate function $I_T(\pi^\delta|\gamma)$ as the sum of the contributions on each time interval $[0, \delta]$, $[\delta, 2\delta]$ and $[2\delta, T]$. The first contribution vanishes because π^δ solves the hydrodynamic equation in this interval. On the time interval $[\delta, 2\delta]$, $\partial_t \rho_i^\delta = -\partial_t \tau_{2\delta-t} = -\frac{1}{2} \Delta \tau_{2\delta-t} + \sum_{v \in \mathcal{V}} \tilde{v}[v \cdot \nabla \chi_v(\tau_{2\delta-t})] = -\frac{1}{2} \Delta(\rho_i^\delta, \mathbf{p}_i^\delta) + \sum_{v \in \mathcal{V}} \tilde{v}[v \cdot \nabla \chi_v(\rho_i^\delta, \mathbf{p}_i^\delta)]$. In particular, the second contribution is equal to

$$\sup_{G \in \mathcal{C}_0^{1,2}(\bar{\Omega}_T)} \sum_{i=1}^d \left\{ \int_0^\delta ds \int_{D^d} du \partial_{u_i}(\rho, \mathbf{p}) \cdot \partial_{u_i} G - \sum_{v \in \mathcal{V}} \int_0^\delta dt \int_{D^d} du \chi_v(\rho, \mathbf{p}) [\tilde{v} \cdot \partial_{u_i} G]^2 \right\}$$

which, by Lemma 6.5 is bounded from above, and therefore this last expression converges to zero as $\delta \downarrow 0$. Finally, the third contribution is bounded by $I_T(\pi|\gamma)$ because π^δ in this interval is just a time translation of the path π . □

Recall the definition of the set \mathfrak{U} given at the ending of Sect. 2.2. Let Π_2 be the set of all paths π in Π_1 with the property that for every $\delta > 0$ there exists $\epsilon > 0$ such that, for $k = 0, \dots, d$, $d(\pi_t^k(\cdot), \partial \mathfrak{U}) \geq \epsilon$ for all $t \in [\delta, T]$, where $\partial \mathfrak{U}$ stands for the boundary of \mathfrak{U} .

We begin by proving an auxiliary lemma.

Lemma 5.2 *Let $\pi, \lambda \in \mathfrak{U}$, and let $\pi^\epsilon = (1 - \epsilon)\pi + \epsilon\lambda$, $0 \leq \epsilon \leq 1$. Then, for all $v \in \mathcal{V}$, we have*

$$\theta_v(\Lambda(\pi^\epsilon)) = (1 - \epsilon)\theta_v(\Lambda(\pi)) + \epsilon\theta_v(\Lambda(\lambda)).$$

Proof Fix some $\lambda \in \mathfrak{U}$. Observe that

$$\left(\sum_{v \in \mathcal{V}} \theta_v(\Lambda(\lambda)), \sum_{v \in \mathcal{V}} v_1 \theta_v(\Lambda(\lambda)), \dots, \sum_{v \in \mathcal{V}} v_d \theta_v(\Lambda(\lambda)) \right) = (\lambda_0, \lambda_1, \dots, \lambda_d)$$

is a linear system with $d + 1$ equations and $|\mathcal{V}|$ unknowns (given by $\theta_v(\Lambda(\lambda))$, for $v \in \mathcal{V}$). Therefore, any solution of this linear system can be expressed as a linear combination of λ_i , $i = 0, 1, \dots, d$. The proof follows from this fact. □

Remark 5.3 In the particular case when $d = 1$ and the set of velocities is $\mathcal{V} = \{v, -v\} \subset \mathbb{R}$, a simple computation gives the unique solution

$$\theta_v(\Lambda(\lambda_0, \lambda_1)) = \frac{\lambda_0}{2} + \frac{\lambda_1}{2v} \quad \text{and} \quad \theta_{-v}(\Lambda(\lambda_0, \lambda_1)) = \frac{\lambda_0}{2} - \frac{\lambda_1}{2v}.$$

Lemma 5.4 *The set Π_2 is $I_T(\cdot|\gamma)$ -dense.*

Proof By Lemma 5.1, it is enough to show that each path $\pi(t, du) = (\rho, \mathbf{p})(t, u)du$ in Π_1 can be approximated by paths in Π_2 . Fix π in Π_1 and let τ be as in the proof of the previous lemma. For each $0 < \varepsilon < 1$, let $(\rho^\varepsilon, \mathbf{p}^\varepsilon) = (1 - \varepsilon)(\rho, \mathbf{p}) + \varepsilon\tau$, $\pi^\varepsilon(t, du) = (\rho^\varepsilon, \mathbf{p}^\varepsilon)(t, u)du$. Note that $\mathcal{Q}(\pi^\varepsilon) < \infty$ because \mathcal{Q} is convex and both $\mathcal{Q}(\pi)$ and $\mathcal{Q}(\tau)$ are finite. Hence, π^ε belongs to D_γ since both ρ and τ satisfy the boundary conditions. Moreover, It is clear that π^ε converges to π as $\varepsilon \downarrow 0$. By the lower semicontinuity of $I_T(\cdot|\gamma)$, in order to conclude the proof, it is enough to show that

$$\overline{\lim}_{N \rightarrow \infty} I_T(\pi^\varepsilon|\gamma) \leq I_T(\pi|\gamma). \tag{5.1}$$

By Lemma 4.7, there exists $H \in H_0^1(\pi)$ such that (ρ, \mathbf{p}) solves (4.9). Let $P_{i,v}(\pi) = \chi_v(\rho, \mathbf{p})(\tilde{v} \cdot \partial_{u_i} H - v_i)$, and note that $P_{i,v}(\tau) = -v_i \chi_v(\tau)$. Let also

$$P_{i,v}^\varepsilon = (1 - \varepsilon)P_{i,v}(\pi) + \varepsilon P_{i,v}(\tau).$$

Observe that, by Lemma 4.7,

$$I_T(\pi|\gamma) = \frac{1}{4} \|H\|_\pi^2,$$

and that, using the definition of $\|\cdot\|_\pi$ in (4.8),

$$\begin{aligned} \frac{1}{4} \|H\|_\pi^2 &= \frac{1}{4} \sum_{v \in \mathcal{V}} \sum_{i=1}^d \int_0^T dt \int_{D^d} du \chi_v(\rho, \mathbf{p})(\tilde{v} \cdot \partial_{u_i} H)^2 \\ &= \frac{1}{4} \sum_{v \in \mathcal{V}} \sum_{i=1}^d \int_0^T dt \int_{D^d} du \frac{(P_{i,v}(\pi) + v_i \chi_v(\rho, \mathbf{p}))^2}{\chi_v(\rho, \mathbf{p})}. \end{aligned}$$

A simple computation shows that

$$\begin{aligned} \mathbb{J}_G(\pi^\varepsilon) &= \sum_{v \in \mathcal{V}} \sum_{i=1}^d \int_0^T \int_{D^d} [P_{i,v}^\varepsilon + \chi_v(\rho^\varepsilon, \mathbf{p}^\varepsilon)v_i](\tilde{v} \cdot \partial_{u_i} G) - \chi_v(\rho^\varepsilon, \mathbf{p}^\varepsilon)(\tilde{v} \cdot \partial_{u_i} G)^2 \\ &= \frac{1}{4} \sum_{v \in \mathcal{V}} \sum_{i=1}^d \int_0^T dt \int_{D^d} du \frac{[P_{i,v}^\varepsilon + \chi_v(\rho^\varepsilon, \mathbf{p}^\varepsilon)v_i]^2}{\chi_v(\rho^\varepsilon, \mathbf{p}^\varepsilon)} \\ &\quad - \left(\frac{1}{2} \frac{P_{i,v}^\varepsilon + \chi_v(\rho^\varepsilon, \mathbf{p}^\varepsilon)}{\sqrt{\chi_v(\rho^\varepsilon, \mathbf{p}^\varepsilon)}} - \sqrt{\chi_v(\rho, \mathbf{p})}(\tilde{v} \cdot \partial_{u_i} G) \right)^2. \end{aligned}$$

Let

$$A_\varepsilon = \frac{1}{4} \sum_{v \in \mathcal{V}} \sum_{i=1}^d \int_0^T dt \int_{D^d} du \frac{[P_{i,v}^\varepsilon + \chi_v(\rho^\varepsilon, \mathbf{p}^\varepsilon)v_i]^2}{\chi_v(\rho^\varepsilon, \mathbf{p}^\varepsilon)},$$

and

$$B_\varepsilon(G) = \int_0^T dt \int_{D^d} du \left(\frac{1}{2} \frac{P_{i,v}^\varepsilon + \chi_v(\rho^\varepsilon, \mathbf{p}^\varepsilon)}{\sqrt{\chi_v(\rho^\varepsilon, \mathbf{p}^\varepsilon)}} - \sqrt{\chi_v(\rho, \mathbf{p})}(\tilde{v} \cdot \partial_{u_i} G) \right).$$

This implies that

$$I_T(\pi^\epsilon | \gamma) = \sup_G \mathbb{J}_G(\pi^\epsilon) = \sup_G \{A_\epsilon - B_\epsilon(G)^2\} = A_\epsilon - \inf_G B_\epsilon(G)^2 \leq A_\epsilon,$$

where the supremum and infimum are taken over in G in $C_c^\infty(\Omega_T) \times [C_c^\infty(D^d)]^d$.

It remains to be shown that A_ϵ is uniformly integrable in ϵ . However, this is a simple consequence of Lemma 5.2. □

Let Π be the subset of Π_2 consisting of all those paths π which are solutions of (4.9) for some $H \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})$.

Theorem 5.5 *The set Π is $I_T(\cdot | \gamma)$ -dense.*

Proof By the previous lemma, it is enough to show that each path π in Π_2 can be approximated by paths in Π . Fix $\pi(t, du) = (\rho, \mathbf{p})(t, u)du$ in Π_2 . By Lemma 4.7, there exists $H \in H_0^1(\pi)$ such that (ρ, \mathbf{p}) solves (4.9). Since π belongs to $\Pi_2 \subset \Pi_1$, (ρ, \mathbf{p}) is the weak solution of (3.2) in some time interval $[0, 2\delta]$ for some $\delta > 0$. In particular, $\tilde{v} \cdot \partial_{u_i} H = 0$ a.e in $[0, 2\delta] \times D^d, i = 1, \dots, d, v \in \mathcal{V}$. This implies, by (2.1), that $\nabla H^k = 0$ a.e. in $[0, 2\delta] \times D^d, k = 0, \dots, d$. On the other hand, since π belongs to Π_1 , there exists $\epsilon > 0$ such that, for $k = 0, \dots, d, d(\pi_t^k(\cdot), \partial \mathbb{L}) \geq \epsilon$ for $\delta \leq t \leq T$. Therefore,

$$\int_0^T dt \int_{D^d} \|\nabla H_t^k(u)\|^2 du < \infty, \quad k = 0, \dots, d. \tag{5.2}$$

Since H belongs to $H_0^1(\pi)$, there exists a sequence of functions $\{H^n = (H^{n,1}, \dots, H^{n,d}); n \geq 1\}$ in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$ converging to H in $H_0^1(\pi)$. We may assume of course that $\nabla H_t^{n,k} \equiv 0$ in the time interval $[0, \delta], k = 0, \dots, d$. In particular,

$$\lim_{n \rightarrow \infty} \int_0^T dt \int_{D^d} du \|\nabla H_t^{n,k}(u) - \nabla H_t^k(u)\|^2 = 0, \quad k = 0, \dots, d. \tag{5.3}$$

For each integer $n > 0$, let (ρ^n, \mathbf{p}^n) be the weak solution of (4.9) with H^n in place of H and set $\pi^n(t, du) = (\rho^n, \mathbf{p}^n)(t, u)du$. By (4.10) and since χ is bounded above in $[0, 1]$ by $1/4$, we have that

$$\begin{aligned} I_T(\pi^n | \gamma) &= \frac{1}{2} \sum_{v \in \mathcal{V}} \sum_{i=1}^d \int_0^T dt \langle \chi_v(\rho_i^n, \mathbf{p}_i^n), (\tilde{v} \cdot \partial_{u_i} H_t^n)^2 \rangle_2 \\ &\leq C_0 \sum_{v \in \mathcal{V}} \sum_{i=1}^d \int_0^T dt \int_{D^d} du (\tilde{v} \cdot \partial_{u_i} H_t^n(u))^2. \end{aligned}$$

In particular, by (5.2) and (5.3), $I_T(\pi^n | \gamma)$ is uniformly bounded on n . Thus, by Theorem 4.6, the sequence π^n is relatively compact in $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$.

The sequence π^n has a subsequence converging to some π^0 in $D([0, T], \mathcal{M}^0)$. To keep notation simple, we will assume that the sequence π^n converges to π^0 . For every G in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$,

$$\begin{aligned}
 & \int_{D^d} G(T, u) \cdot (\rho_t^n, \mathbf{p}_t^n)(T, u) du - \int_{D^d} G(0, u) \cdot \gamma(u) du \\
 &= \int_0^T dt \int_{D^d} du \left\{ (\rho_t^n, \mathbf{p}_t^n)(t, u) \cdot \partial_t G(t, u) + \frac{1}{2} (\rho_t^n, \mathbf{p}_t^n)(t, u) \cdot \sum_{i=1}^d \partial_{u_i}^2 G(t, u) \right\} \\
 & \quad - \frac{1}{2} \int_0^T dt \int_{\{1\} \times \mathbb{T}^{d-1}} dSb(\tilde{u}) \cdot \partial_{u_1} G(t, u) + \frac{1}{2} \int_0^T dt \int_{\{0\} \times \mathbb{T}^{d-1}} dSa(\tilde{u}) \cdot \partial_{u_1} G(t, u) \\
 & \quad - \int_0^T dt \int_{D^d} du \sum_{v \in \mathcal{V}} \tilde{v} \cdot \chi_v(\rho_t^n, \mathbf{p}_t^n) \sum_{i=1}^d v_i \partial_{u_i} G(t, u) \\
 & \quad + \sum_{v \in \mathcal{V}} \sum_{i=1}^d \int_0^T dt \int_{D^d} du \chi_v(\rho_t^n, \mathbf{p}_t^n) [\tilde{v} \cdot \partial_{u_i} H^n][\tilde{v} \cdot \partial_{u_i} G].
 \end{aligned}$$

Letting $n \rightarrow \infty$ in this equation, we obtain the same equation with π^0 and H in place of π^n and H^n , respectively, if

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \int_0^T dt \int_{D^d} du \sum_{v \in \mathcal{V}} \tilde{v} \cdot \chi_v(\rho_t^n, \mathbf{p}_t^n) \sum_{i=1}^d v_i \partial_{u_i} G(t, u) \\
 &= \int_0^T dt \int_{D^d} du \sum_{v \in \mathcal{V}} \tilde{v} \cdot \chi_v(\rho_t^0, \mathbf{p}_t^0) \sum_{i=1}^d v_i \partial_{u_i} G(t, u), \\
 & \lim_{n \rightarrow \infty} \sum_{v \in \mathcal{V}} \sum_{i=1}^d \int_0^T dt \int_{D^d} du \chi_v(\rho_t^n, \mathbf{p}_t^n) [\tilde{v} \cdot \partial_{u_i} H^n][\tilde{v} \cdot \partial_{u_i} G] \\
 &= \sum_{v \in \mathcal{V}} \sum_{i=1}^d \int_0^T dt \int_{D^d} du \chi_v(\rho_t^0, \mathbf{p}_t^0) [\tilde{v} \cdot \partial_{u_i} H][\tilde{v} \cdot \partial_{u_i} G].
 \end{aligned} \tag{5.4}$$

We prove the second claim, the first one being simpler. Note first that we can replace H^n by H in the previous limit, because χ is bounded in $[0, 1]$ by $1/4$, and (5.3) holds. Now, (ρ^n, \mathbf{p}^n) converges to (ρ^0, \mathbf{p}^0) weakly in $L^2(\Omega_T) \times [L^2(D^d)]^d$ because π^n converges to π^0 in $D([0, T], \mathcal{M}^0)$. Since $I_T(\pi^n | \gamma)$ is uniformly bounded, by Corollary 4.4 and Lemma 4.5, (ρ^n, \mathbf{p}^n) converges to (ρ^0, \mathbf{p}^0) strongly in $L^2(\Omega_T) \times [L^2(D^d)]^d$ which implies (5.4). In particular, since (5.2) holds, by uniqueness of weak solutions of (4.9), $\pi^0 = \pi$ and we are done. □

6 Large Deviations

We prove in this section Theorem 2.1, which is the dynamical large deviations principle for the empirical measure of boundary driven stochastic lattice gas model with many conserved quantities. The proof uses some of the ideas introduced in [9].

6.1 Superexponential Estimates

It is well known that one of the main steps in the derivation of the upper bound is a super-exponential estimate which allows the replacement of local functions by functionals of the empirical density in the large deviations regime.

Let κ be as in the beginning of Sect. 4. Note that since ν_κ^N is not the invariant state, there are no reasons for $\langle -N^2 \mathcal{L}_N f, f \rangle_{\nu_\kappa^N}$ to be positive. The next statement shows that this expression is almost positive.

For each function $f : X_N \rightarrow \mathbb{R}$, let $D_{\nu_\kappa^N}(f)$ be

$$D_{\nu_\kappa^N}(f) = D_{\nu_\kappa^N}^{ex}(f) + D_{\nu_\kappa^N}^c(f) + D_{\nu_\kappa^N}^b(f),$$

where

$$D_{\nu_\kappa^N}^{ex}(f) = \sum_{v \in \mathcal{V}} \sum_{x \in D_N^d} \sum_{x+z \in D_N^d} P_N(z-x, v) \int [\sqrt{f(\eta^{x,z,v})} - \sqrt{f(\eta)}]^2 \nu_\kappa^N(d\eta),$$

$$D_{\nu_\kappa^N}^c(f) = \sum_{q \in \mathcal{Q}} \sum_{x \in D_N^d} \int p(x, q, \eta) [\sqrt{f(\eta^{x,q})} - \sqrt{f(\eta)}]^2 \nu_\kappa^N(d\eta),$$

and

$$\begin{aligned} D_{\nu_\kappa^N}^b(f) &= \sum_{v \in \mathcal{V}} \sum_{x \in \{1\} \times \mathbb{T}_N^{d-1}} \int [\alpha_v(\tilde{x}/N)(1 - \eta(x, v)) + (1 - \alpha_v(\tilde{x}/N))\eta(x, v)] \\ &\quad \times [\sqrt{f(\sigma^{x,v}\eta)} - \sqrt{f(\eta)}]^2 \nu_\kappa^N(d\eta) \\ &\quad + \sum_{v \in \mathcal{V}} \sum_{x \in \{N-1\} \times \mathbb{T}_N^{d-1}} \int [\beta_v(\tilde{x}/N)(1 - \eta(x, v)) + (1 - \beta_v(\tilde{x}/N))\eta(x, v)] \\ &\quad \times [\sqrt{f(\sigma^{x,v}\eta)} - \sqrt{f(\eta)}]^2 \nu_\kappa^N(d\eta). \end{aligned}$$

Proposition 6.1 *There exist constants $C_1 > 0$ and $C_2 = C_2(\alpha, \beta) > 0$ such that for every density f with respect to ν_κ^N , we have*

$$\langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_\kappa^N} \leq -C_1 D_{\nu_\kappa^N}(f) + C_2 N^{d-2}.$$

The proof of this proposition is elementary and is thus omitted.

Further, we may choose κ for which there exists a constant $\theta > 0$ such that:

$$\begin{aligned} \kappa(u_1, \tilde{u}) &= d(0, \tilde{u}) \quad \text{if } 0 \leq u_1 \leq \theta, \\ \kappa(u_1, \tilde{u}) &= d(1, \tilde{u}) \quad \text{if } 1 - \theta \leq u_1 \leq 1, \end{aligned}$$

for all $\tilde{u} \in \mathbb{T}^{d-1}$. In that case, for every N large enough, ν_κ^N is reversible for the process with generator \mathcal{L}_N^b and then $\langle -N^2 \mathcal{L}_N^b f, f \rangle_{\nu_\kappa^N}$ is positive.

Fix $L \geq 1$ and a configuration η , let $\mathbf{I}^L(x, \eta) := \mathbf{I}^L(x) = (I_0^L(x), \dots, I_d^L(x))$ be the average of the conserved quantities in a cube of the length L centered at x :

$$\mathbf{I}^L(x) = \frac{1}{|\Lambda_L|} \sum_{z \in x + \Lambda_L} \mathbf{I}(\eta_z),$$

where, $\Lambda_L = \{-L, \dots, L\}^d$ and $|\Lambda_L| = (2L + 1)^d$ is the discrete volume of box Λ_L .

For each $G \in \mathcal{C}(\overline{\Omega_T}) \times [C(\overline{D^d})]^d$, and each $\varepsilon > 0$, let

$$V_{N\varepsilon}^{G,1}(s, \eta) = \frac{1}{N^d} \sum_{k=0}^d \sum_{i,j=1}^d \sum_{x \in D_N^d} \partial_{u_i} G^k(s, x/N) [\tau_x \tilde{V}_{N\varepsilon}^{j,k}],$$

where

$$\begin{aligned} \tilde{V}_{N\varepsilon}^{j,k}(\eta) &= \frac{1}{(2\ell + 1)^d} \sum_{y \in \Lambda_{N\varepsilon}} \sum_{v \in \mathcal{V}} v_k \sum_{z \in \mathbb{Z}^d} p(z, v) z_j \tau_y(\eta(0, v)[1 - \eta(z, v)]) \\ &\quad - \sum_{v \in \mathcal{V}} v_j v_k \chi_v(\mathbf{I}^\ell(0)), \end{aligned}$$

and let

$$\begin{aligned} V_{N\varepsilon}^{G,2}(s, \eta) &= \frac{1}{2N^d} \sum_{v \in \mathcal{V}} \sum_{x \in D_N^d} \sum_{i=1}^d \sum_{j,k=0}^d v_k v_j \partial_{u_i}^N G_i^j(x/N) \partial_{u_i}^N G_i^k(x/N) \\ &\quad \times \{ \eta(x, v)[1 - \eta(x + e_i, v)] + \eta(x, v)[1 - \eta(x - e_i, v)] - 2\chi_v(\mathbf{I}^\ell(0)) \}. \end{aligned}$$

Let, again, $G \in \mathcal{C}(\overline{\Omega_T}) \times [C(\overline{D^d})]^d$, and consider the quantities

$$V_N^-(s, \eta, G) = \frac{1}{N^{d-1}} \sum_{k=0}^d \sum_{\tilde{x} \in \mathbb{T}_N^{d-1}} G_k(s, \tilde{x}/N) \left(I_k(\eta_{(1, \tilde{x})}(s)) - \sum_{v \in \mathcal{V}} v_k \alpha_v(\tilde{x}/N) \right),$$

and

$$V_N^+(s, \eta, G) = \frac{1}{N^{d-1}} \sum_{k=0}^d \sum_{\tilde{x} \in \mathbb{T}_N^{d-1}} G_k(s, \tilde{x}/N) \left(I_k(\eta_{(N-1, \tilde{x})}(s)) - \sum_{v \in \mathcal{V}} v_k \beta_v(\tilde{x}/N) \right).$$

Proposition 6.2 Fix $G \in \mathcal{C}(\overline{\Omega_T}) \times [C(\overline{D^d})]^d$, H in $\mathcal{C}([0, T] \times \Gamma) \times [C(\Gamma)]^d$, a cylinder function Ψ and a sequence $\{\eta^N; N \geq 1\}$ of configurations with η^N in X_N . For every $\delta > 0$,

$$\begin{aligned} \overline{\lim}_{\varepsilon \rightarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \log \mathbb{P}_{\eta^N} \left[\left| \int_0^T V_{N\varepsilon}^{G,j}(s, \eta_s) ds \right| > \delta \right] &= -\infty, \\ \overline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \mathbb{P}_{\eta^N} \left[\left| \int_0^T V_N^\pm(s, \eta, G) \right| > \delta \right] &= -\infty, \end{aligned}$$

for $j = 1, 2$.

The proof of the above proposition follows from Proposition 6.1, the replacement lemmas proved in [12], and the computation presented in [3], p. 78, for nonreversible processes.

For each $\varepsilon > 0$ and π in $\mathcal{M}_+ \times \mathcal{M}^d$, for $k = 0, \dots, d$, denote by $\Xi_\varepsilon(\pi_k) = \pi_k^\varepsilon$ the absolutely continuous measure obtained by smoothing the measure π_k :

$$\Xi_\varepsilon(\pi_k)(dx) = \pi_k^\varepsilon(dx) = \frac{1}{U_\varepsilon} \frac{\pi_k(\mathbf{\Lambda}_\varepsilon(x))}{|\mathbf{\Lambda}_\varepsilon(x)|} dx,$$

where $\Lambda_\varepsilon(x) = \{y \in D^d; |y - x| \leq \varepsilon\}$, $|A|$ stands for the Lebesgue measure of the set A , and $\{U_\varepsilon; \varepsilon > 0\}$ is a strictly decreasing sequence converging to 1: $U_\varepsilon > 1$, $U_\varepsilon > U_{\varepsilon'}$ for $\varepsilon > \varepsilon'$, $\lim_{\varepsilon \downarrow 0} U_\varepsilon = 1$. Let

$$\pi^{N,\varepsilon} = (\Xi_\varepsilon(\pi_0^N), \Xi_\varepsilon(\pi_1^N), \dots, \Xi_\varepsilon(\pi_d^N)).$$

A simple computation shows that $\pi^{N,\varepsilon}$ belongs to \mathcal{M}^0 for N sufficiently large because $U_\varepsilon > 1$, and that for each continuous function $H : D^d \rightarrow \mathbb{R}^{d+1}$,

$$\langle \pi^{N,\varepsilon}, H \rangle = \frac{1}{N^d} \sum_{x \in D_N^d} H(x/N) \cdot I^{\varepsilon N}(x) + O(N, \varepsilon),$$

where $O(N, \varepsilon)$ is absolutely bounded by $C_0\{N^{-1} + \varepsilon\}$ for some finite constant C_0 depending only on H .

For each H in $\mathfrak{C}_0^{1,2}(\overline{\Omega_T})$ consider the exponential martingale M_t^H defined by

$$M_t^H = \exp \left\{ N^d \left[\langle \pi_t^N, H_t \rangle - \langle \pi_0^N, H_0 \rangle - \frac{1}{N^d} \int_0^t e^{-N^d \langle \pi_s^N, H_s \rangle} (\partial_s + N^2 \mathcal{L}_N) e^{N^d \langle \pi_s^N, H_s \rangle} ds \right] \right\}.$$

Recall from Sect. 2.2 the definition of the functional \hat{J}_H . An elementary computation shows that

$$M_T^H = \exp \{ N^d [\hat{J}_H(\pi^{N,\varepsilon}) + \mathbb{V}_{N,\varepsilon}^H + c_H^1(\varepsilon) + c_H^2(N^{-1})] \}. \tag{6.1}$$

In this formula,

$$\begin{aligned} \mathbb{V}_{N,\varepsilon}^H &= - \int_0^T V_{N\varepsilon}^{G,1}(s, \eta) ds - \sum_{i=1}^d \int_0^T V_{N\varepsilon}^{G,2}(s, \eta) ds \\ &\quad + V_N^+(s, \eta, \partial_{u_1} H) - V_N^-(s, \eta, \partial_{u_1} H) + \langle \pi_0^N, H_0 \rangle - \langle \gamma, H_0 \rangle; \end{aligned}$$

and $c_H^j : \mathbb{R}_+ \rightarrow \mathbb{R}$, $j = 1, 2$, are functions depending only on H such that $c_H^j(\delta)$ converges to 0 as $\delta \downarrow 0$. In particular, the martingale M_T^H is bounded by $\exp\{C(H, T)N^d\}$ for some finite constant $C(H, T)$ depending only on H and T . Therefore, Proposition 6.2 holds for $\mathbb{P}_{\eta^N}^H = \mathbb{P}_{\eta^N} M_T^H$ in place of \mathbb{P}_{η^N} .

6.2 Energy Estimates

To exclude paths with infinite energy in the large deviations regime, we need an energy estimate. We state first the following technical result.

Lemma 6.3 *There exists a finite constant C_0 , depending on T , such that for every G in $C_c^\infty(\Omega_T)$, every integer $1 \leq i \leq d$, $0 \leq k \leq d$, and every sequence $\{\eta^N; N \geq 1\}$ of configurations with η^N in X_N ,*

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \log \mathbb{E}_{\eta^N} \left[\exp \left\{ N^d \int_0^T dt \langle \pi_t^{N,k}, \partial_{u_i} G \rangle \right\} \right] \leq C_0 \left\{ 1 + \int_0^T \|G_t\|_2^2 dt \right\}.$$

The proof of this proposition follows from Lemma 3.8 in [12], and the fact that $d\delta_{\eta^N}/d\nu_{\kappa}^N \leq C N^d$, for some positive constant $C = C(\kappa)$.

For each G in $C_c^\infty(\Omega_T)$ and each integer $1 \leq i \leq d$, let $\tilde{Q}_{i,k}^G : D([0, T], \mathcal{M}_+ \times \mathcal{M}^d) \rightarrow \mathbb{R}$ be the function given by

$$\tilde{Q}_{i,k}^G(\pi) = \int_0^T dt \langle \pi_t^k, \partial_{u_i} G_t \rangle - C_0 \int_0^T dt \int_{D^d} du G(t, u)^2.$$

Notice that

$$\sup_{G \in C_c^\infty(\Omega_T)} \{ \tilde{Q}_{i,k}^G(\pi) \} = \frac{Q_{i,k}(\pi)}{4C_0}. \tag{6.2}$$

Fix a sequence $\{G_r; r \geq 1\}$ of smooth functions dense in $L^2([0, T], H^1(D^d))$. For any positive integers m, l , let

$$B_{m,l}^k = \left\{ \pi \in D([0, T], \mathcal{M}_+ \times \mathcal{M}^d); \max_{\substack{1 \leq j \leq m \\ 1 \leq i \leq d}} \tilde{Q}_{i,k}^{G_j}(\pi) \leq l \right\}.$$

Since, for fixed G in $C_c^\infty(\Omega_T)$ and $1 \leq i \leq d$ integer, the function $\tilde{Q}_{i,k}^G$ is continuous, $B_{m,l}$ is a closed subset of $D([0, T], \mathcal{M})$.

Lemma 6.4 *There exists a finite constant C_0 , depending on T , such that for any positive integers r, l and any sequence $\{\eta^N; N \geq 1\}$ of configurations with η^N in X_N ,*

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \log Q_{\eta^N}[(B_{m,l}^k)^c] \leq -l + C_0,$$

where $k = 0, \dots, d$.

Proof For integers $1 \leq k \leq r$ and $1 \leq i \leq d$, by Chebychev inequality and by Lemma 6.3,

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \log \mathbb{P}_{\eta^N}[\tilde{Q}_{i,k}^{G_m} > l] \leq -l + C_0.$$

Hence, from

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \log(a_N + b_N) \leq \max \left\{ \overline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \log a_N, \overline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \log b_N \right\}, \tag{6.3}$$

we obtain the desired inequality. □

Lemma 6.5 *There exists a finite constant C_0 , depending on T , such that for every G in $C_c^\infty(\Omega_T) \times [C_c^\infty(D^d)]^d$, and every sequence $\{\eta^N; N \geq 1\}$ of configurations with η^N in X_N ,*

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \log \mathbb{E}_{\nu_k^N} \left[\exp \left\{ N^d \int_0^T \sum_{i=1}^d \sum_{k=0}^d dt \langle \pi_t^{N,k}, \partial_{u_i} G^k \rangle \right\} \right] \leq C_0 \left\{ 1 + \int_0^T \|G_t\|_\pi^2 dt \right\}.$$

In particular, we have that if (ρ, \mathbf{p}) is the solution of (3.2), then

$$\sup_{G \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})} \sum_{i=1}^d \left\{ \int_0^T ds \int_{D^d} du \partial_{u_i}(\rho, \mathbf{p}) \cdot \partial_{u_i} G - \sum_{v \in \mathcal{V}} \int_0^T dt \int_{D^d} du \chi_v(\rho, \mathbf{p}) [\tilde{v} \cdot \partial_{u_i} G]^2 \right\},$$

is finite, and vanishes if $T \rightarrow 0$.

Proof Applying Feynman-Kac’s formula and using the same arguments of Lemma 3.3 in [12], we have that

$$\frac{1}{N^d} \log E_{v_\kappa^N} \left[\exp \left\{ N \int_0^T ds \sum_{i=1}^d \sum_{k=0}^d \sum_{x \in D_N^d} (I_k(\eta_x(s)) - I_k(\eta_{x-e_i}(s))) \partial_{u_i} G^k(s, x/N) \right\} \right]$$

is bounded above by

$$\frac{1}{N^d} \int_0^T \lambda_s^N ds,$$

where λ_s^N is equal to

$$\sup_f \left\{ \left\langle N \sum_{i,k} \sum_{x \in D_N^d} (I_k(\eta(x)) - I_k(\eta(x - e_i))) \partial_{u_i} G^k(s, x/N), f \right\rangle_{v_\kappa^N} + N^2 \langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{v_\kappa^N} \right\},$$

where the supremum is taken over all densities f with respect to v_κ^N . By Proposition 6.1, the expression inside brackets is bounded above by

$$CN^d - \frac{N^2}{2} D_{v_\kappa^N}(f) + \sum_{i,k} \sum_{x \in D_N^d} \left\{ N \partial_{u_i} G^k(s, x/N) \int [I_k(\eta_x) - I_k(\eta_{x-e_i})] f(\eta) v_\kappa^N(d\eta) \right\}.$$

We now rewrite the term inside the brackets as

$$\sum_{v \in \mathcal{V}} \sum_{i=1}^d \sum_{x \in D_N^d} \left\{ \int N(\tilde{v} \cdot \partial_{u_i} G(s, x/N)) [\eta(x, v) - \eta(x - e_i, v)] f(\eta) v_\kappa^N(d\eta) \right\}.$$

Writing $\eta(x, v) - \eta(x - e_i, v) = \eta(x, v)[1 - \eta(x - e_i, v)] - \eta(x - e_i, v)[1 - \eta(x, v)]$, and applying the same arguments in Lemma 3.8 of [12], we obtain that

$$\begin{aligned} & N(\tilde{v} \cdot \partial_{u_i} G(s, x/N)) \int [\eta(x, v) - \eta(x - e_i, v)] f(\eta) v_\kappa^N(d\eta) \\ & \leq (\tilde{v} \cdot \partial_{u_i} G(s, x/N))^2 \int \eta(x, v)[1 - \eta(x - e_i, v)] f(\eta^{x-e_i, x, v}) dv_\kappa^N \\ & \quad + \frac{1}{4} \int f(\eta^{x-e_i, x, v}) \left[N \left(1 - \frac{\gamma_{x-e_i, v}}{\gamma_{x, v}} \right) \right]^2 v_\kappa^N(d\eta) \\ & \quad + N^2 \int \frac{1}{2} [\sqrt{f(\eta^{x-e_i, x, v})} - \sqrt{f(\eta)}]^2 v_\kappa^N(d\eta) \\ & \quad + 2(\tilde{v} \cdot \partial_{u_i} G(s, x/N))^2 \int \eta(x, v)[1 - \eta(x - e_i, v)] (\sqrt{f(\eta)}) \\ & \quad + \sqrt{f(\eta^{x-e_i, x, v})}^2 v_\kappa^N(d\eta), \end{aligned}$$

we have that $(\sqrt{f(\eta)} + \sqrt{f(\eta^{x-e_i, x, v})})^2 \leq 2(f(\eta) + f(\eta^{x-e_i, x, v}))$. An application of the replacement lemma (Lemma 3.7 in [12]) concludes the proof. \square

6.3 Upper Bound

Fix a sequence $\{F_j; j \geq 1\}$ of smooth functions dense in $C(\overline{D^d})$ for the uniform topology, with positive coordinates. For $j \geq 1$ and $\delta > 0$, let

$$D_{j,\delta} = \left\{ \pi \in D([0, T], \mathcal{M}_+ \times \mathcal{M}^d); |\langle \pi_t^k, F_j \rangle| \leq \check{v}^k |\mathcal{V}| \int_{D^d} F_j(x) dx + C_j \delta, k = 0, \dots, d, 0 \leq t \leq T \right\},$$

where $\check{v}^0 = 1$ and $\check{v}^k = \check{v}$, $C_j = \|\nabla F_j\|_\infty$ and ∇F is the gradient of F . Clearly, the set $D_{j,\delta}$, $j \geq 1$, $\delta > 0$, is a closed subset of $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$. Moreover, if

$$E_{m,\delta} = \bigcap_{j=1}^m D_{j,\delta},$$

we have that $D([0, T], \mathcal{M}^0) = \bigcap_{n \geq 1} \bigcap_{m \geq 1} E_{m,1/n}$. Note, finally, that for all $m \geq 1$, $\delta > 0$,

$$\pi^{N,\varepsilon} \text{ belongs to } E_{m,\delta} \text{ for } N \text{ sufficiently large.} \tag{6.4}$$

Fix a sequence of configurations $\{\eta^N; N \geq 1\}$ with η^N in X_N and such that $\pi^N(\eta^N)$ converges to $\gamma(u)du$ in $\mathcal{M}_+ \times \mathcal{M}^d$. Let A be a subset of $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$,

$$\frac{1}{N^d} \log \mathbb{P}_{\eta^N}[\pi^N \in A] = \frac{1}{N^d} \log \mathbb{E}_{\eta^N}[M_T^H (M_T^H)^{-1} \mathbf{1}\{\pi^N \in A\}].$$

Maximizing over π^N in A , we get from (6.1) that the last term is bounded above by

$$-\inf_{\pi \in A} \hat{J}_H(\pi^\varepsilon) + \frac{1}{N^d} \log \mathbb{E}_{\eta^N}[M_T^H e^{-N^d \mathcal{V}_{N,\varepsilon}^H}] - c_H^1(\varepsilon) - c_H^2(N^{-1}).$$

Since $\pi^N(\eta^N)$ converges to $\gamma(u)du$ in $\mathcal{M}_+ \times \mathcal{M}^d$ and since Proposition 6.2 holds for $\mathbb{P}_{\eta^N}^H = \mathbb{P}_{\eta^N} M_T^H$ in place of \mathbb{P}_{η^N} , the second term of the previous expression is bounded above by some $C_H(\varepsilon, N)$ such that

$$\overline{\lim}_{\varepsilon \rightarrow 0} \overline{\lim}_{N \rightarrow \infty} C_H(\varepsilon, N) = 0.$$

Hence, for every $\varepsilon > 0$, and every H in $\mathfrak{C}_0^{1,2}(\overline{\Omega_T})$,

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \log \mathbb{P}_{\eta^N}[A] \leq -\inf_{\pi \in A} \hat{J}_H(\pi^\varepsilon) + C'_H(\varepsilon), \tag{6.5}$$

where $\lim_{\varepsilon \rightarrow 0} C'_H(\varepsilon) = 0$. Let

$$B_{r,l} = \left\{ \pi \in D([0, T], \mathcal{M}_+ \times \mathcal{M}^d); \max_{\substack{1 \leq j \leq r \\ 1 \leq i \leq d}} \sum_{k=0}^d \tilde{Q}_{i,k}^{G_j}(\pi) \leq l \right\},$$

and, for each $H \in \mathfrak{C}_0^{1,2}(\overline{\Omega_T})$, each $\varepsilon > 0$ and any $r, l, m, n \in \mathbb{Z}_+$, let $J_{H,\varepsilon}^{r,l,m,n} : D([0, T], \mathcal{M}_+ \times \mathcal{M}^d) \rightarrow \mathbb{R} \cup \{\infty\}$ be the functional given by

$$J_{H,\varepsilon}^{r,l,m,n}(\pi) = \begin{cases} \hat{J}_H(\pi^\varepsilon) & \text{if } \pi \in B_{r,l} \cap E_{m,1/n}, \\ +\infty & \text{otherwise.} \end{cases}$$

This functional is lower semicontinuous because so is $\hat{J}_H \circ \Xi_\varepsilon$ and because $B_{r,l}, E_{m,1/n}$ are closed subsets of $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$.

Let \mathcal{O} be an open subset of $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$. By Lemma 6.4, (6.3), (6.4) and (6.5),

$$\begin{aligned} & \overline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \log Q_{\eta^N}[\mathcal{O}] \\ & \leq \max \left\{ \overline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \log Q_{\eta^N}[\mathcal{O} \cap B_{r,l} \cap E_{m,1/n}], \overline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \log Q_{\eta^N}[(B_{r,l})^c] \right\} \\ & \leq \max \left\{ - \inf_{\pi \in \mathcal{O} \cap B_{r,l} \cap E_{m,1/n}} \hat{J}_H(\pi^\varepsilon) + C'_H(\varepsilon), -l + C_0 \right\} \\ & = - \inf_{\pi \in \mathcal{O}} L_{H,\varepsilon}^{r,l,m,n}(\pi), \end{aligned}$$

where

$$L_{H,\varepsilon}^{r,l,m,n}(\pi) = \min \{ J_{H,\varepsilon}^{r,l,m,n}(\pi) - C'_H(\varepsilon), l - C_0 \}.$$

In particular,

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \log Q_{\eta^N}[\mathcal{O}] \leq - \sup_{H,\varepsilon,r,l,m,n} \inf_{\pi \in \mathcal{O}} L_{H,\varepsilon}^{r,l,m,n}(\pi).$$

Note that, for each $H \in \mathfrak{C}_0^{1,2}(\overline{\Omega_T})$, each $\varepsilon > 0$ and $r, l, m, n \in \mathbb{Z}_+$, the functional $L_{H,\varepsilon}^{r,l,m,n}$ is lower semicontinuous. Then, by Lemma A2.3.3 in [10], for each compact subset \mathcal{K} of $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$,

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \log Q_{\eta^N}[\mathcal{K}] \leq - \inf_{\pi \in \mathcal{K}} \sup_{H,\varepsilon,r,l,m,n} L_{H,\varepsilon}^{r,l,m,n}(\pi).$$

By (6.2) and since $D([0, T], \mathcal{M}^0) = \bigcap_{n \geq 1} \bigcap_{m \geq 1} E_{m,1/n}$,

$$\begin{aligned} & \overline{\lim}_{\varepsilon \rightarrow 0} \overline{\lim}_{l \rightarrow \infty} \overline{\lim}_{r \rightarrow \infty} \overline{\lim}_{m \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} L_{H,\varepsilon}^{r,l,m,n}(\pi) \\ & = \begin{cases} \hat{J}_H(\pi) & \text{if } Q(\pi) < \infty \text{ and } \pi \in D([0, T], \mathcal{M}^0), \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

This result and the last inequality imply the upper bound for compact sets because \hat{J}_H and J_H coincide on $D([0, T], \mathcal{M}^0)$. To pass from compact sets to closed sets, we have to obtain exponential tightness for the sequence $\{Q_{\eta^N}\}$. This means that there exists a sequence of compact sets $\{\mathcal{K}_n; n \geq 1\}$ in $D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$ such that

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N^d} \log Q_{\eta^N}(\mathcal{K}_n^c) \leq -n.$$

The proof presented in [2] for the non interacting zero range process is easily adapted to our context.

6.4 Lower Bound

The proof of the lower bound is similar to the one in the convex periodic case. We just sketch it and refer to [10, Sect. 10.5]. Fix a path π in Π and let $H \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})$ be such that π is the weak solution of (4.9). Recall from the previous section the definition of the martingale M_t^H and denote by $\mathbb{P}_{\eta^N}^H$ the probability measure on $D([0, T], X_N)$ given by $\mathbb{P}_{\eta^N}^H[A] = \mathbb{E}_{\eta^N}[M_T^H \mathbf{1}\{A\}]$. Under $\mathbb{P}_{\eta^N}^H$ and for each $0 \leq t \leq T$, the empirical measure π_t^N converges in probability to π_t . Further,

$$\lim_{N \rightarrow \infty} \frac{1}{N^d} H(\mathbb{P}_{\eta^N}^H | \mathbb{P}_{\eta^N}) = I_T(\pi | \gamma),$$

where $H(\mu | \nu)$ stands for the relative entropy of μ with respect to ν . From these two results we can obtain that for every open set $\mathcal{O} \subset D([0, T], \mathcal{M}_+ \times \mathcal{M}^d)$ which contains π ,

$$\liminf_{N \rightarrow \infty} \frac{1}{N^d} \log \mathbb{P}_{\eta^N}[\mathcal{O}] \geq -I_T(\pi | \gamma).$$

The lower bound follows from this and the $I_T(\cdot | \gamma)$ -density of Π established in Theorem 5.5.

Acknowledgement We would like to thank Claudio Landim for suggesting this problem.

References

1. Beltrán, J., Landim, C.: A lattice gas model for the incompressible Navier-Stokes equation. *Ann. Inst. Henri Poincaré, Probab. Stat.* **44**, 886–914 (2008)
2. Benois, O.: Large deviations for the occupation times of independent particle systems. *Ann. Appl. Probab.* **6**, 269–296 (1996)
3. Benois, O., Kipnis, C., Landim, C.: Large deviations from the hydrodynamical limit of mean zero asymmetric zero range processes. *Stoch. Process. Appl.* **55**, 65–89 (1995)
4. Bernardin, C.: Stationary nonequilibrium properties for a heat conduction model. *Phys. Rev. E* **78**, 021134 (2008)
5. Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G., Landim, C.: Large deviation approach to non equilibrium processes in stochastic lattice gases. *Bull. Braz. Math. Soc.* **37**, 611–643 (2006)
6. Bertini, L., De Sole, A., Gabrielli, G., Jona-Lasinio, G., Landim, C.: Stochastic interacting particle systems out of equilibrium. *J. Stat. Mech. Theory Exp.* P07014 (2007)
7. Derrida, B.: Non equilibrium steady states: fluctuations and large deviations of the density and of the current. *J. Stat. Mech. Theory Exp.* P07023 (2007)
8. Esposito, R., Marra, R., Yau, H.T.: Navier-Stokes equations for stochastic particle systems on the lattice. *Commun. Math. Phys.* **182**, 395–456 (1996)
9. Farfan, J., Landim, C., Mourragui, M.: Hydrostatics and dynamical large deviations of boundary driven gradient symmetric exclusion. Preprint [0903.5526](#)
10. Kipnis, C., Landim, C.: *Scaling Limits of Interacting Particle Systems*. Springer, Berlin (1999)
11. Quastel, J., Yau, H.T.: Lattice gases, large deviations, and the incompressible Navier-Stokes equations. *Ann. Math.* **148**, 51–108 (1998)
12. Simas, A.B.: Hydrodynamic limit for a boundary driven stochastic lattice gas model with many conserved quantities. *J. Stat. Phys.* **139**, 219–251 (2010)
13. Simon, J.: Compact sets in the space $L^p(0, T; B)$. *Ann. Mat. Pura Appl.* **CXLVI**, 65–96 (1987)
14. Spohn, H.: Long range correlations for stochastic lattice gases in a non-equilibrium steady state. *J. Stat. Phys. A, Math. Gen.* **16**, 4275–4291 (1983)
15. Zeidler, E.: *Nonlinear Functional Analysis and Its Applications. II/A, Linear Monotone Operators and II/B Nonlinear Monotone Operators*. Springer, Berlin (1980)